

# Optimal asymmetric $1 \rightarrow 4$ quantum cloning in arbitrary dimension

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**Abstract.** We present an optimal asymmetric  $1 \rightarrow 4$  quantum cloner. Our derivation generalizes the constructions of optimal asymmetric  $1 \rightarrow 2$  and  $1 \rightarrow 3$  quantum cloners in [Quantum Inf. Comput **5**, 583 (2005)]. We explicitly prove the optimality of this cloner and give the maximum achievable fidelities. We also present the relation between the optimal quantum cloner with the multipartite entangled state which shows the singlet monogamy inequality in [Phys. Rev. Lett. **103**, 050501 (2009)].

In quantum mechanics, the perfect copying of an unknown state is forbidden as a result of the linear superposition principle [1]. However, it is possible to construct approximate quantum cloning machines (QCM) yielding high-fidelity copies. In 1996, Bužek and Hillery [2] constructed the optimal symmetric  $1 \rightarrow 2$  universal QCM for qubits, soon afterwards its generalization to optimal symmetric  $m \rightarrow n$  universal QCMs for both qubits and arbitrary-dimensional systems were given [3–8]. All the above results were on symmetric cloning where all the copies have the same fidelity, however, from a more practical point especially in eavesdropping attack on certain kinds of QKD protocols [9–11], it is more interesting to consider optimal asymmetric cloning where the copies may have different fidelities. The family of optimal  $1 \rightarrow 2$  asymmetric UQCMs for arbitrary-dimensional systems has been fully characterized [12–14]. However, for a long time, the proposed universal asymmetric cloning machines have only been conjectured to be optimal, and the proof of the optimality has been provided not long before [15–17]. In [17], Fiurášek et al. proved the optimality of universal asymmetric  $1 \rightarrow 2$  cloning machines for qudits, they further extended the concept to quantum triplicators and constructed optimal asymmetric  $1 \rightarrow 3$  cloning machine. In this paper, we will generalize their proof to an optimal asymmetric  $1 \rightarrow 4$  cloning machine. In [18], Kay et al. related the optimal cloning with the singlet monogamy inequality in a multipartite system. In fact, the constraints on the shareability of maximal entanglement between multiple qudits is inherently linked with constraints on producing multiple copies of an unknown state. The singlet monogamy inequality in [18] was deduced from the best ansatz state for optimal asymmetric cloning. Here, we also

discuss the relations between their results and our optimal asymmetric  $1 \rightarrow 4$  cloning machine. It should also be noted that, besides the theoretical developments on optimal quantum cloning, experimental implementation of various cloning machines has also been realized [19–23].

The organization of this paper is outlined as follows. Firstly, we formalize the general  $1 \rightarrow n$  asymmetric cloning problem in terms of the Jamiołkowski isomorphism. Then we briefly review the known optimal asymmetric  $1 \rightarrow 2$  and  $1 \rightarrow 3$  cloning machines. In the following, we make a through discussion on our optimal asymmetric  $1 \rightarrow 4$  cloner and its relation with the singlet monogamy presented in reference [18]. Finally, we give a brief conclusion.

In order to formalize the general  $1 \rightarrow n$  asymmetric cloning problem, we first introduce the Jamiołkowski isomorphism which gives a one-to-one correspondence between a completely positive (CP) map  $\mathcal{S}$  from  $\mathcal{H}_{in}$  to  $\mathcal{H}_{out}$  and a positive operator  $S$  on  $\mathcal{H}_{in} \otimes \mathcal{H}_{out}$ . Let qudit 1 be the input and qudits  $1, 2, \dots, n$  be the output copies. Then cloning operations correspond to completely positive maps  $\mathcal{S}$  transforming qudit 1's input state into output cloned states on qudits  $1, 2, \dots, n$ . By introducing a reference qudit 0, we can construct Jamiołkowski isomorphism between completely positive maps  $\mathcal{S}$  and positive semidefinite operators  $S \geq 0$ ,

$$S = I \otimes \mathcal{S}(d\Phi_{01}^+), \quad (1)$$

here,  $d$  is the qudit's dimension and  $\Phi^+ = |\Phi^+\rangle\langle\Phi^+|$ ,  $|\Phi^+\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle|j\rangle$ . Notice that the positive semidefinite operator  $S$  operates on qudits  $0, 1, \dots, n$ . Since qudit 0 is a reference qudit, the completely positive map  $\mathcal{S}$  should have no effect on it. At the same time, we assume that the completely positive map  $\mathcal{S}$  is trace preserving.

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Therefore the positive semidefinite operator  $S$  should satisfy the constraints  $\text{Tr}_{1,\dots,n}[S] = I_0$  and  $\text{Tr}[S] = d$ .

In terms of the above isomorphism, we can formalize the fidelities of the general universal asymmetric cloning as follows. For an arbitrary input state  $|\psi\rangle$ , qudit  $k$ 's fidelity is  $F_k(\psi) = \text{Tr}[\psi_0^T \otimes I_{1,\dots,\hat{k},\dots,n} \otimes \psi_k \cdot S]$ . Here,  $\psi = |\psi\rangle\langle\psi|$  and  $\hat{k}$  represents the exclusion of  $k$ . Because we are considering universal cloning machines which clone equally well all states from the input Hilbert space, we should calculate the averaged fidelity  $F_k = \int_{\psi} F_k(\psi) d\psi = \text{Tr}[S \cdot L_k]$ . Here  $L_k$  is given as

$$\begin{aligned} L_k &= \int_{\psi} \psi_0^T \otimes I_{1,\dots,\hat{k},\dots,n} \otimes \psi_k d\psi \\ &= \frac{1}{d(d+1)} [I_{0,1,\dots,n} + d\Phi_{0k}^+ \otimes I_{1,\dots,\hat{k},\dots,n}] \\ &= \frac{1}{d(d+1)} [I_{0,1,\dots,n} + d\tilde{L}_k] \end{aligned} \quad (2)$$

here  $\tilde{L}_k = \Phi_{0k}^+ \otimes I_{1,\dots,\hat{k},\dots,n}$ . The optimal asymmetric cloning machine  $S$  should maximize a convex mixture of each clone's fidelities  $F = \sum_{k=1}^n p_k F_k$ , where  $p_k \geq 0$  and  $\sum_{k=1}^n p_k = 1$ . Inserting (2) into  $F$ , we obtain  $F = \text{Tr}[S \cdot L]$ , where  $L = \sum_{k=1}^n p_k L_k = \frac{1}{d(d+1)} [I_{0,1,\dots,n} + d\tilde{L}]$ ,  $\tilde{L} = \sum_{k=1}^n p_k \tilde{L}_k$ . Obviously,  $\tilde{L}$ 's maximum eigenvalue upper bounds the optimal fidelity  $F$ . If we can find out  $\tilde{L}$ 's eigenspace with the maximum eigenvalue, the projection onto this space will provide us with the optimal asymmetric cloning machine  $S$ . For  $n = 2, 3$ , Fiurášek et al. [17] have analyzed  $\tilde{L}$ 's support space and given  $S$  based on  $\tilde{L}$ 's maximum eigenvalue. Before giving our optimal cloner for  $n = 4$ , we will first give a brief review on these two cases.

*Optimal asymmetric 1 → 2 cloning machine.* We have  $\tilde{L} = p\tilde{L}_1 + (1-p)\tilde{L}_2$ , its support space has dimension  $2d$  and can be divided into two  $d$ -fold degenerate spaces. The eigenstates have the following form

$$|\lambda_j; k\rangle = \beta_1 |\Phi^+\rangle_{01} |k\rangle_2 + \beta_2 |\Phi^+\rangle_{02} |k\rangle_1$$

where  $j = 1, 2$  and  $k = 0, \dots, d-1$ . The two eigenvalues  $\lambda_1 > \lambda_2$  are roots of the quadratic equation

$$\lambda^2 - \lambda + p(1-p)[1-d^{-2}] = 0.$$

The optimal cloning transformation  $S$  is the projector onto the  $d$ -dimensional subspace spanned by the eigenstates  $|\lambda_1; k\rangle$  corresponding to the maximum eigenvalue  $\lambda_1$ ,

$$S = \sum_{k=0}^{d-1} |\lambda_1; k\rangle\langle\lambda_1; k|$$

*Optimal asymmetric 1 → 3 cloning machine.* We have  $\tilde{L} = p_1\tilde{L}_1 + p_2\tilde{L}_2 + p_3\tilde{L}_3$ , its support space has dimension  $3d^2$  and has six different eigenvalues. Three of them are

$d(d+1)/2$ -fold degenerate and the corresponding eigenstates read,

$$\begin{aligned} |\lambda_+; kl\rangle &= \beta_1 |\Phi^+\rangle_{01} |kl^+\rangle_{23} + \beta_2 |\Phi^+\rangle_{02} |kl^+\rangle_{13} \\ &\quad + \beta_3 |\Phi^+\rangle_{03} |kl^+\rangle_{12} \end{aligned}$$

with  $l \geq k$ . Here, we take  $|kl^+\rangle = (|kl\rangle + |lk\rangle)/\sqrt{2}$  if  $k \neq l$ , while  $|kk^+\rangle = |kk\rangle$ . The three eigenvalues ( $\lambda_{+,1} > \lambda_{+,2} > \lambda_{+,3}$ ) can be determined as roots of the cubic equation

$$\begin{aligned} P_+(\lambda_+) &= \lambda_+^3 - \lambda_+^2 + \lambda_+(p_1p_2 + p_1p_3 + p_2p_3)(1-d^{-2}) \\ &\quad - p_1p_2p_3(1+2d^{-3} - 3d^{-2}) = 0. \end{aligned} \quad (3)$$

The other three eigenvalues are  $d(d-1)/2$ -fold degenerate and their corresponding eigenstates read,

$$\begin{aligned} |\lambda_-; kl\rangle &= \beta_1 |\Phi^+\rangle_{01} |kl^-\rangle_{23} + \beta_2 |\Phi^+\rangle_{02} |kl^-\rangle_{13} \\ &\quad + \beta_3 |\Phi^+\rangle_{03} |kl^-\rangle_{12} \end{aligned} \quad (4)$$

where  $|kl^-\rangle = (|kl\rangle - |lk\rangle)/\sqrt{2}$ . They can be determined as roots of the cubic equation

$$\begin{aligned} P_-(\lambda_-) &= \lambda_-^3 - \lambda_-^2 + \lambda_-(p_1p_2 + p_1p_3 + p_2p_3)(1-d^{-2}) \\ &\quad - p_1p_2p_3(1-2d^{-3} - 3d^{-2}) = 0. \end{aligned} \quad (5)$$

As pointed out in [17], the difference between  $P_+$  and  $P_-$  is a vertical shift of  $4p_1p_2p_3/d^3$ , hence the largest root of  $P_+$  is greater than  $P_-$ . The optimal trace-preserving 1 → 3 cloning map can then be expressed as the properly normalized projector onto the subspace by the  $d(d+1)/2$  eigenstates with eigenvalue  $\lambda_{+,1}$ ,

$$S = \frac{2}{d+1} \sum_{l \geq k} |\lambda_{+,1}; kl\rangle\langle\lambda_{+,1}; kl|$$

*Optimal asymmetric 1 → 4 cloner.* Now, we have  $\tilde{L} = p_1\tilde{L}_1 + p_2\tilde{L}_2 + p_3\tilde{L}_3 + p_4\tilde{L}_4$ , its support space has dimension  $4d^3$ . We will show that  $\tilde{L}$  has only eight different eigenvalues, four of them are  $d^2(d+1)/2$ -fold degenerate and the other four are  $d^2(d-1)/2$ -fold degenerate. The former's eigenstates can be written as,

$$\begin{aligned} |\lambda_+; kl, m\rangle &= \beta_1 |\Phi^+\rangle_{01} |kl, m\rangle_{234} + \beta_2 |\Phi^+\rangle_{02} |kl, m\rangle_{134} \\ &\quad + \beta_3 |\Phi^+\rangle_{03} |kl, m\rangle_{124} + \beta_4 |\Phi^+\rangle_{04} |kl, m\rangle_{123} \end{aligned} \quad (6)$$

where  $|kl, m\rangle_{xyz}$  is a properly normalized symmetric state: for  $k = l = m$ ,  $|kl, m\rangle_{xyz} = |k\rangle_x |k\rangle_y |k\rangle_z$ ; for  $k = m \neq l$ ,  $|kl, m\rangle_{xyz} = (|kl^+\rangle_{xy} |k\rangle_z + |kl^+\rangle_{xz} |k\rangle_y + |kl^+\rangle_{yz} |k\rangle_x)/\sqrt{6}$ ; for  $k \neq l \neq m$ ,  $|kl, m\rangle_{xyz} = (|kl^+\rangle_{xy} |m\rangle_z + |kl^+\rangle_{xz} |m\rangle_y + |kl^+\rangle_{yz} |m\rangle_x)/\sqrt{3}$ . The four eigenvalues ( $\lambda_{+,1} > \lambda_{+,2} >$

$\lambda_{+,3} > \lambda_{+,4}$ ) are the roots of the following equation,

$$\begin{aligned} Q_+(\lambda_+) &= \lambda_+^4 - \lambda_+^3 + \lambda_+^2(p_1p_2 + p_1p_3 + p_1p_4 + p_2p_3 \\ &\quad + p_2p_4 + p_3p_4)(1 - d^{-2}) \\ &\quad - \lambda_+(p_1p_2p_3 + p_1p_2p_4 + p_1p_3p_4 + p_2p_3p_4) \\ &\quad \times (1 + 2d^{-3} - 3d^{-2}) \\ &\quad + p_1p_2p_3p_4(1 - 6d^{-2} + 8d^{-3} - 3d^{-4}) \\ &= 0. \end{aligned}$$

The coefficients  $\beta_1, \beta_2, \beta_3, \beta_4$  can be expressed in terms of  $p_1, p_2, p_3, p_4$  and  $\lambda_+$  by solving the system of linear equations,

$$\begin{aligned} (\lambda_+ - p_1)\beta_1 - \frac{p_1}{d}(\beta_2 + \beta_3 + \beta_4) &= 0 \\ (\lambda_+ - p_2)\beta_2 - \frac{p_2}{d}(\beta_1 + \beta_3 + \beta_4) &= 0 \\ (\lambda_+ - p_3)\beta_3 - \frac{p_3}{d}(\beta_1 + \beta_2 + \beta_4) &= 0 \\ (\lambda_+ - p_4)\beta_4 - \frac{p_4}{d}(\beta_1 + \beta_2 + \beta_3) &= 0. \end{aligned}$$

The normalization of the eigenstates  $|\lambda_+; kl, m\rangle$  imposes the constraint

$$\begin{aligned} \beta_1^2 + \beta_2^2 + \beta_3^2 + \beta_4^2 + \frac{2}{d}(\beta_1\beta_2 + \beta_1\beta_3 + \beta_1\beta_4 + \beta_2\beta_3 \\ + \beta_2\beta_4 + \beta_3\beta_4) = 1. \end{aligned} \quad (7)$$

The eigenstates of the  $d^2(d-1)/2$ -fold degenerate eigenvalues can be properly constructed as follows,

$$\begin{aligned} |\lambda_-; kl, m\rangle &= \beta_1|\Phi^+\rangle_{01} (|kl^-\rangle_{23}|m\rangle_4 - |kl^-\rangle_{24}|m\rangle_3 \\ &\quad + |kl^-\rangle_{34}|m\rangle_2) + \beta_2|\Phi^+\rangle_{02} (|kl^-\rangle_{13}|m\rangle_4 - |kl^-\rangle_{14}|m\rangle_3 \\ &\quad + |kl^-\rangle_{34}|m\rangle_1) + \beta_3|\Phi^+\rangle_{03} (-|kl^-\rangle_{12}|m\rangle_4 - |kl^-\rangle_{24}|m\rangle_1 \\ &\quad + |kl^-\rangle_{14}|m\rangle_2) + \beta_4|\Phi^+\rangle_{04} (|kl^-\rangle_{23}|m\rangle_1 + |kl^-\rangle_{12}|m\rangle_3 \\ &\quad - |kl^-\rangle_{13}|m\rangle_2). \end{aligned} \quad (8)$$

It can be checked that the four eigenvalues  $\lambda_-$  of these states are the roots of the following equation,

$$\begin{aligned} Q_-(\lambda_-) &= \lambda_-^4 - \lambda_-^3 + \lambda_-^2(p_1p_2 + p_1p_3 + p_1p_4 + p_2p_3 \\ &\quad + p_2p_4 + p_3p_4)(1 - d^{-2}) \\ &\quad - \lambda_-(p_1p_2p_3 + p_1p_2p_4 + p_1p_3p_4 + p_2p_3p_4) \\ &\quad \times (1 - 2d^{-3} - 3d^{-2}) \\ &\quad + p_1p_2p_3p_4(1 - 6d^{-2} - 8d^{-3} - 3d^{-4}) \\ &= 0. \end{aligned}$$

The difference between  $Q_+$  and  $Q_-$  is

$$\begin{aligned} Q_-(\lambda) - Q_+(\lambda) &= 4d^{-3}(p_1p_2p_3 + p_1p_2p_4 + p_1p_3p_4\lambda \\ &\quad + p_2p_3p_4) - 16d^{-3}p_1p_2p_3p_4. \end{aligned}$$

Now we will show that the largest root of  $Q_+(\lambda) = 0$  is greater than  $Q_-(\lambda) = 0$ , i.e.,  $\lambda_{+,1} > \lambda_{-,1}$ . First we notice the following two facts: (a) for large  $\lambda$ ,  $Q_+(\lambda) > 0$ ; (b) for  $\lambda > p_1 = \max\{p_1, p_2, p_3, p_4\}$ , we have  $Q_-(\lambda) - Q_+(\lambda) > 0$ . Here, without loss of generality we assume that  $p_1$  is the largest probability. Now, if  $\lambda_{+,1} > p_1$ , then combing the above two facts it can be easily deduced that  $\lambda_{+,1} > \lambda_{-,1}$ . In fact,  $\lambda_{+,1} > p_1$  really holds. To show this we only need  $Q_+(p_1) < 0$ . Through direct calculations and rearrangement, we obtain

$$\begin{aligned} Q_+(p_1) &= -d^{-2}p_1[(p_1 - p_3)p_2(p_1 - p_4) + (p_1 - p_2)(p_1 - p_3)p_4 \\ &\quad + (p_1 - p_2)p_3(p_1 - p_4)] - 2d^{-3}p_1[(p_1 - p_4)p_2p_3 \\ &\quad + (p_1 - p_3)p_2p_4 + (p_1 - p_2)p_3p_4] - 3d^{-4}p_1p_2p_3p_4. \end{aligned}$$

Obviously  $Q_+(p_1) < 0$  since  $p_1$  is the largest probability.

The optimal trace-preserving 1 → 4 cloning map can now be expressed as the properly normalized projector onto the  $d^2(d+1)/2$ -fold degenerate subspace with eigenvalue  $\lambda_{+,1}$ ,

$$S = \frac{2}{d(d+1)} \sum_{\substack{l \geq k \\ m}} |\lambda_{+,1}; kl, m\rangle \langle \lambda_{+,1}; kl, m|,$$

where the prefactor originates from the constraint  $\text{Tr}(S) = d$ . A unitary implementation of this map requires three ancillary qudits 5, 6, 7, and can be characterized by the purification of  $S$ ,

$$|\Phi\rangle = C \sum_{k \leq l; m} |\lambda_+; kl, m\rangle_{01234} |kl^+\rangle_{56} |m\rangle_7,$$

where  $C$  is a proper normalization constant.

We can express the fidelities in terms of the coefficients  $\beta_1, \beta_2, \beta_3, \beta_4$ , by noting that

$$\begin{aligned} \langle \lambda_+; kl, m | \Phi_{01}^+ \otimes I_{234} | \lambda_+; kl, m \rangle &= (\beta_1 + \beta_2/d + \beta_3/d \\ &\quad + \beta_4/d)^2 \\ \langle \lambda_+; kl, m | \Phi_{02}^+ \otimes I_{134} | \lambda_+; kl, m \rangle &= (\beta_1/d + \beta_2 + \beta_3/d \\ &\quad + \beta_4/d)^2 \\ \langle \lambda_+; kl, m | \Phi_{03}^+ \otimes I_{124} | \lambda_+; kl, m \rangle &= (\beta_1/d + \beta_2/d \\ &\quad + \beta_3 + \beta_4/d)^2 \\ \langle \lambda_+; kl, m | \Phi_{04}^+ \otimes I_{123} | \lambda_+; kl, m \rangle &= (\beta_1/d + \beta_2/d + \beta_3/d \\ &\quad + \beta_4)^2. \end{aligned}$$

Using the normalization condition (7), we obtain the optimal fidelities,

$$\begin{aligned} F_1 &= 1 - \frac{d-1}{d} \left[ \beta_2^2 + \beta_3^2 + \beta_4^2 + \frac{2(\beta_2\beta_3 + \beta_2\beta_4 + \beta_3\beta_4)}{d+1} \right], \\ F_2 &= 1 - \frac{d-1}{d} \left[ \beta_1^2 + \beta_3^2 + \beta_4^2 + \frac{2(\beta_1\beta_3 + \beta_1\beta_4 + \beta_3\beta_4)}{d+1} \right], \\ F_3 &= 1 - \frac{d-1}{d} \left[ \beta_1^2 + \beta_2^2 + \beta_4^2 + \frac{2(\beta_1\beta_2 + \beta_1\beta_4 + \beta_2\beta_4)}{d+1} \right], \\ F_4 &= 1 - \frac{d-1}{d} \left[ \beta_1^2 + \beta_2^2 + \beta_3^2 + \frac{2(\beta_1\beta_2 + \beta_1\beta_3 + \beta_2\beta_3)}{d+1} \right]. \end{aligned} \quad (9)$$

This provides a parametric description of the whole class of the optimal universal asymmetric  $1 \rightarrow 4$  cloning machines in a Hilbert space of arbitrary dimension  $d$ . It should also be noted that the constraint (7) gives a trade-off relation on the cloning fidelities  $F_i$ .

To show our optimal asymmetric cloning explicitly, we next present some examples. We first consider a trivial case: there is no cloning at all, but we just transfer the original state to a different site. Without lose of generality, we consider the input state is in site 1 so we have the fidelity  $F_1 = 1$ . Since our cloning machine is a universal cloner, we know nothing about the input state, when all input state is in site 1, other sites contain no information about the input. The reduced density operators in sites 2, 3, 4 should be the same one which is the identity with fidelity  $\frac{1}{d}$  for input. Really we find in (9) and (7) the solution  $F_1 = 1, F_2 = F_3 = F_4 = \frac{1}{d}$ . We next consider the symmetric universal cloning machine with equal fidelities. From equation (7) with equal parameters, we find  $\beta_1 = \beta_2 = \beta_3 = \beta_4 = \frac{1}{2\sqrt{3+d}}$ , so the optimal fidelities can be found to be,  $F_1 = F_2 = F_3 = F_4 = \frac{d+7}{4(d+1)}$ . The optimal fidelity [3–8] of the symmetric universal cloning is already known,  $F = (m(d+n) + (n-m))/(d+m)n$ , for case  $1 \rightarrow 4, m = 1, n = 4$ , we have  $F = \frac{d+7}{4(d+1)}$ . Really we find  $F = F_j, j = 1, 2, 3, 4$ . So we can recover the symmetric optimal cloning machine from our asymmetric one. By simply setting one or two parameters  $\beta_j = 0$ , we also recover the  $1 \rightarrow 3$  and  $1 \rightarrow 2$  cases of asymmetric cloners. We notice the interesting fact that the unobjected qudits have fidelities greater than  $\frac{1}{d}$  which means that we still have some information about the input state besides an optimal  $1 \rightarrow 2$  or  $1 \rightarrow 3$  cloner. This can be understood by noticing in  $1 \rightarrow 3$  and  $1 \rightarrow 2$  cloning, the ancillas (anti-clone, see [24]) carry some information about the input which contributes to the unobjected qudits in  $1 \rightarrow 4$  cloning [17].

In [18], Kay et al. introduced a singlet monogamy inequality based on optimal universal asymmetric  $1 \rightarrow n$  cloning. There they regard the maximally entangled state  $|\Phi^+\rangle$  as the resource to transport the cloning quantum information on qudit 0, then qudit  $k$ 's cloning fidelity is determined by its singlet entanglement with qudit 0. The singlet monogamy comes after a multipartite entangled

state,

$$|\Psi\rangle_n = \sum_{k=1}^n \beta_k |\Phi^+\rangle_{0k} |\Phi\rangle_{1,\dots,\hat{k},\dots,n}, \quad (10)$$

where  $|\Phi\rangle$  is the normalized uniform superposition over all permutations of  $|\Phi^+\rangle^{\otimes(n-1)/2}$  for odd  $n$ , and  $|\Phi^+\rangle^{\otimes(n-2)/2}|0\rangle$  for even  $n$ . Besides the normalization condition, the coefficients  $\{\beta_k\}$  maximize a matrix determined by  $\{p_k\}$  [18]. It should be noted that these  $\beta_k$ s are completely the same with equation (6). However, state (10) has only been proposed to be the best ansatz state for asymmetric  $1 \rightarrow n$  cloning and an exact proof is lack. Here, we will show that for  $n = 4$ , state  $|\Psi\rangle_4$  lies in the maximal eigenvalue space and it indeed provides the optimal asymmetric cloning. First we write explicitly  $|\Psi\rangle_4$  as the following,

$$\begin{aligned} |\Psi\rangle_4 &= N \{ \beta_1 |\Phi^+\rangle_{01} [ |\Phi^+\rangle_{23}|0\rangle_4 + |\Phi^+\rangle_{24}|0\rangle_3 + |\Phi^+\rangle_{34}|0\rangle_2 ] \\ &\quad + \beta_2 |\Phi^+\rangle_{02} [ |\Phi^+\rangle_{13}|0\rangle_4 + |\Phi^+\rangle_{14}|0\rangle_3 + |\Phi^+\rangle_{34}|0\rangle_1 ] \\ &\quad + \beta_3 |\Phi^+\rangle_{03} [ |\Phi^+\rangle_{12}|0\rangle_4 + |\Phi^+\rangle_{14}|0\rangle_2 + |\Phi^+\rangle_{24}|0\rangle_1 ] \\ &\quad + \beta_4 |\Phi^+\rangle_{04} [ |\Phi^+\rangle_{12}|0\rangle_3 + |\Phi^+\rangle_{13}|0\rangle_2 + |\Phi^+\rangle_{23}|0\rangle_1 ] \} \end{aligned}$$

here  $N$  is a normalization constant. In order to get  $|\Psi\rangle_4$  from our maximal eigenvalue space  $\{|\lambda_+; kl, m\rangle\}$ , we start from the following unnormalized eigenstates,

$$\begin{aligned} |\lambda_+; kl, m\rangle_{unnorm} &= \beta_1 |\Phi^+\rangle_{01} (|kl^+\rangle_{23}|m\rangle_4 \\ &\quad + |kl^+\rangle_{24}|m\rangle_3 + |kl^+\rangle_{34}|m\rangle_2) \\ &\quad + \beta_2 |\Phi^+\rangle_{02} (|kl^+\rangle_{13}|m\rangle_4 \\ &\quad + |kl^+\rangle_{14}|m\rangle_3 + |kl^+\rangle_{34}|m\rangle_1) \\ &\quad + \beta_3 |\Phi^+\rangle_{03} (|kl^+\rangle_{12}|m\rangle_4 \\ &\quad + |kl^+\rangle_{24}|m\rangle_1 + |kl^+\rangle_{14}|m\rangle_2) \\ &\quad + \beta_4 |\Phi^+\rangle_{04} (|kl^+\rangle_{23}|m\rangle_1 \\ &\quad + |kl^+\rangle_{12}|m\rangle_3 + |kl^+\rangle_{13}|m\rangle_2). \end{aligned}$$

From them we can construct a purified state by adding three ancilla qudits,

$$\begin{aligned} |\Phi\rangle &= C \{ \beta_1 |\Phi^+\rangle_{01} [ (|\Phi^+\rangle_{25} |\Phi^+\rangle_{36} + |\Phi^+\rangle_{26} |\Phi^+\rangle_{35}) |\Phi^+\rangle_{47} \\ &\quad + (|\Phi^+\rangle_{25} |\Phi^+\rangle_{46} + |\Phi^+\rangle_{26} |\Phi^+\rangle_{45}) |\Phi^+\rangle_{37} \\ &\quad + (|\Phi^+\rangle_{35} |\Phi^+\rangle_{46} + |\Phi^+\rangle_{36} |\Phi^+\rangle_{45}) |\Phi^+\rangle_{27} ] \\ &\quad + \beta_2 |\Phi^+\rangle_{02} [ (|\Phi^+\rangle_{15} |\Phi^+\rangle_{36} + |\Phi^+\rangle_{16} |\Phi^+\rangle_{35}) |\Phi^+\rangle_{47} \\ &\quad + (|\Phi^+\rangle_{15} |\Phi^+\rangle_{46} + |\Phi^+\rangle_{16} |\Phi^+\rangle_{45}) |\Phi^+\rangle_{37} \\ &\quad + (|\Phi^+\rangle_{35} |\Phi^+\rangle_{46} + |\Phi^+\rangle_{36} |\Phi^+\rangle_{45}) |\Phi^+\rangle_{17} ] \\ &\quad + \beta_3 |\Phi^+\rangle_{03} [ (|\Phi^+\rangle_{15} |\Phi^+\rangle_{26} + |\Phi^+\rangle_{16} |\Phi^+\rangle_{25}) |\Phi^+\rangle_{47} \\ &\quad + (|\Phi^+\rangle_{25} |\Phi^+\rangle_{46} + |\Phi^+\rangle_{26} |\Phi^+\rangle_{45}) |\Phi^+\rangle_{17} \\ &\quad + (|\Phi^+\rangle_{15} |\Phi^+\rangle_{46} + |\Phi^+\rangle_{16} |\Phi^+\rangle_{45}) |\Phi^+\rangle_{27} ] \\ &\quad + \beta_4 |\Phi^+\rangle_{04} [ (|\Phi^+\rangle_{25} |\Phi^+\rangle_{36} + |\Phi^+\rangle_{26} |\Phi^+\rangle_{35}) |\Phi^+\rangle_{17} \\ &\quad + (|\Phi^+\rangle_{15} |\Phi^+\rangle_{26} + |\Phi^+\rangle_{16} |\Phi^+\rangle_{25}) |\Phi^+\rangle_{37} \\ &\quad + (|\Phi^+\rangle_{15} |\Phi^+\rangle_{36} + |\Phi^+\rangle_{16} |\Phi^+\rangle_{35}) |\Phi^+\rangle_{27} ] \} \end{aligned}$$

where  $C$  is the normalization constant and we have used the identity,

$$\frac{2}{d} \sum_{l \geq k} |kl^+\rangle_{23} |kl^+\rangle_{56} = |\Phi^+\rangle_{25} |\Phi^+\rangle_{36} + |\Phi^+\rangle_{26} |\Phi^+\rangle_{35}.$$

Now, to obtain  $|\Psi\rangle_4$  we simply need to make two projections, the first one is to project onto  $|0\rangle_7$  and the second one is onto  $|\Phi^+\rangle_{56}$ .

In conclusion, we have constructed a universal asymmetric  $1 \rightarrow 4$  cloner and given an explicit proof on its optimality. Its optimal fidelities are parametrically given. Based on our construction, we also derive the multipartite entangled state for singlet monogamy. It may be tempting to further generalize our construction to higher  $1 \rightarrow n$  cloners which are easy in terms of state (10). However, the difficult point is to prove its optimality. We have tried for  $n = 5$  but failed to construct proper eigenstates as equation (4), (8). Therefore, for further generalization some new techniques are needed.

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