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Biquaternionic Proca-type generalization of gravity

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Abstract. In this paper, a biquaternionic model is introduced so as to formulate the generalization of gravity both covering the Proca-type and gravitomagnetic monopole terms. Similarly, the gravitational wave equation including the terms mentioned above is obtained in compact and elegant manner. Moreover, the most generalized form of homogeneous Klein-Gordon equation has been developed for the particle carrying gravitational masses.

1 Introduction

Quaternions are an extension of complex numbers and were invented by Hamilton [1] in 1893. The biquaternion term is used for an eight-dimensional mathematical structure which is composed of two quaternions. By using the quaternion formalism, many alternative formulations in different areas of physics can be derived. Specially in electromagnetic theory, the classical Maxwell equations can be transcribed in the simple and compact form. The origin of this formalism may be stretched to Maxwell himself. Although, in his famous book *Treatise on Electricity and Magnetism* [2], Maxwell used 3-dimensional vector representations to formulate electromagnetism, he also provided their quaternionic forms in a number of places.

Although there is still no direct evidence of their existence, Dirac's magnetic monopole theory [3] is one of the extensions of classical Maxwell's equations. By introducing magnetic monopole terms, a fully symmetric form is obtained between the electric and magnetic fields. In the relevant theory, the quaternions analysis has also been employed to derive Dirac-Maxwell equations in the presence of electric and magnetic sources. As a consequence of Dirac's magnetic monopole theory, the existence of gravitomagnetic mass for the gravity can be proposed to be analogous to the magnetic charge in electromagnetism. Singh [4] has expressed the quaternionic form of the linear gravitational field equations with Heavisidian monopoles. Using the extended form of biquaternionic Maxwell's equations, Majernik [5] has generalized Maxwell-like gravitational field equations by replacing positive-charge terms in Maxwell's equations with negative gravitational mass terms. In another study, Rajput [6] has showed the structural symmetry between the generalized gravitational field and the generalized electromagnetic field associated with gravitomagnetic mass and magnetic monopole terms, respectively. In relevant papers [7–10], Negi and co-workers have used the biquaternion formalism to reformulate the unified theory of linear gravity and electromagnetism in the presence of both magnetic and gravitomagnetic monopoles.

The hypothesis of a non-zero rest mass of the photon leads to further generalization of the classical Maxwell equations. Within this framework, the electrodynamics of Maxwell equations can be extended and generalized further into Proca equations [11]. Similarly, in the case when the gravitons possess non-zero rest mass, an interesting result analogous to the Proca equation is obtained. Argyris and Ciubotariu [12] have shown that the usual Maxwell's equations for electromagnetism can be transformed into Proca-type gravitational field equations with the addition of the terms due to the finite mass of graviton. In electromagnetic theory, Christianto *et al.* have argued the biquaternionic extension of the Klein-Gordon equation [13] and its numerical solution [14]. They have also discussed the Proca equations in terms of biquaternions for an alternative description of superconductivity, via extending London equations [15].

Hyperbolic quaternions share similar properties with ordinary quaternions except their non-associative structure. By using hyperbolic quaternion formalism, Demir *et al.* [16] have also expressed the generalized Proca-Maxwell equations which include magnetic monopole terms in compact, simpler and elegant forms.

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In contrast to biquaternions, octonions with eight dimensions are non-commutative and non-associative mathematical structures. Split octonions and the hyperbolic octonions differ from the classical octonions as they have a hyperbolic basis instead of the complex ones. Candemir *et al.* [17] have expressed the Proca-Maxwell equations in terms of the hyperbolic octonions. Similarly, Bisht and co-workers [18,19] have proposed split octonions to formulate the unified theory of linear gravity and electromagnetism with the simultaneous existence of electric, magnetic, gravitational and Heavisidian charges. Both hyperbolic octonions and split octonios are subalgebra of sixteen dimensional sedenions. Köplinger has examined sedenions in the gravity [20] and gravitoelectromagnetism [21].

Compared to quaternions and octonions, the mathematical structure of the Clifford algebra provides more useful and more elegant formulations in gravity and electromagnetism. By using the complex Clifford algebra formalism $\bar{\boldsymbol{C}}_{3,0}$, Ulrych [22] has introduced a linear-vector model based on hyperbolic numbers for gravitoelectromagnetism. In another paper related to the hyperbolic number formalism in spacetime algebra, Cafaro and Ali [23] have reformulated the Proca-type massive electromagnetism with magnetic monopoles.

Although Argyris and Ciubotariu have proposed Proca-type generalization of gravity based on the non-zero rest mass of gravitons and obtained a result analogous to massive electromagnetism, in the relevant papers, the biquaternionic formulation of this generalization is absent. In this study, we have combined the gravitomagnetic monopole terms with the Proca-type generalization of gravity and expressed them in elegant, economical and compact form. In this formalism, we have shown that all field equations of gravity can be transcribed as one biquaternionic equation. The gravitational wave equation with Proca-type generalization has been obtained in compact and elegant manner. In the next stage, the most generalized form of Klein-Gordon equation has been developed for the particle carrying gravitational masses.

2 Preliminaries

Biquaternions are defined as the complex combination of two quaternions. For two real quaternions $\mathbf{q} = q_0 + q_1\boldsymbol{e}_1 + q_2\boldsymbol{e}_2 + q_3\boldsymbol{e}_3$ and $\mathbf{q}' = q'_0 + q'_1\boldsymbol{e}_1 + q'_2\boldsymbol{e}_2 + q'_3\boldsymbol{e}_3$, a biquaternion $\mathbf{Q}$ is defined as

$$\mathbf{Q} = \mathbf{q} + i\mathbf{q}' = (q_0 + iq'_0) + (q_1 + iq'_1)\boldsymbol{e}_1 + (q_2 + iq'_2)\boldsymbol{e}_2 + (q_3 + iq'_3)\boldsymbol{e}_3. \tag{1a}$$

Here i is the most acknowledged complex unit ($i = \sqrt{-1}$) and components of quaternions $\mathbf{q}$ and $\mathbf{q}'$ are real numbers. Biquaternion $\mathbf{Q}$ is also expressed as

$$\mathbf{Q} = Q_0 + Q_1\boldsymbol{e}_1 + Q_2\boldsymbol{e}_2 + Q_3\boldsymbol{e}_3, \tag{1b}$$

where Q_0, Q_1, Q_2, Q_3 are also complex numbers. Since they have complex numbers, biquaternions are also termed as complex quaternions. A biquaternion can be represented in terms of its scalar and vectors parts as

$$\mathbf{P} = P_0 + \boldsymbol{P}, \tag{2}$$

where P_0 is the scalar and $\boldsymbol{P} = P_1\boldsymbol{e}_1 + P_2\boldsymbol{e}_2 + P_3\boldsymbol{e}_3$ the vector part, respectively. The product of two biquaternions $\mathbf{P}$ and $\mathbf{Q}$ is the same as the real quaternions product,

$$\mathbf{PQ} = (P_0 + \boldsymbol{P})(Q_0 + \boldsymbol{Q}) = P_0Q_0 + P_0\mathbf{Q} + Q_0\boldsymbol{P} - \boldsymbol{P}\cdot\boldsymbol{Q} + \boldsymbol{P}\times\boldsymbol{Q}. \tag{3}$$

The biquaternion product is associative but not commutative. Similar to complex numbers, conjugation is accomplished by changing the sign of the components of the imaginary basis elements,

$$\bar{\mathbf{Q}} = Q_0 - \boldsymbol{Q} = Q_0 - Q_1\boldsymbol{e}_1 - Q_2\boldsymbol{e}_2 - Q_3\boldsymbol{e}_3. \tag{4}$$

The conjugation of the product of two biquaternions satisfies the following equation:

$$(\bar{\mathbf{PQ}}) = \bar{\mathbf{Q}}\bar{\mathbf{P}}. \tag{5}$$

Similarly, the complex conjugate of $\mathbf{Q}$ is also defined as

$$\mathbf{Q}^* = (q_0 - iq'_0) + (q_1 - iq'_1)\boldsymbol{e}_1 + (q_2 - iq'_2)\boldsymbol{e}_2 + (q_3 - iq'_3)\boldsymbol{e}_3. \tag{6}$$

The norm of a biquaternion in general is a complex scalar and is given by

$$N_{\mathrm{Q}} = \mathbf{Q}\bar{\mathbf{Q}} = \bar{\mathbf{Q}}\mathbf{Q} = Q_0^2 + Q_1^2 + Q_2^2 + Q_3^2. \tag{7}$$

Biquaternions with unit norm are termed as unit biquaternions. The inverse of a biquaternion Q, whose norm is non-zero, is presented by

$$\mathbf{Q}^{-1} = \frac{\bar{\mathbf{Q}}}{N_{\mathrm{Q}}}\,. \tag{8}$$

The norm for biquaternions may be zero. Therefore, the biquaternion algebra, unlike the real quaternion algebra, is not a division algebra. While a real quaternion can be used to express the four-dimensional physical quantities in the Euclidean space, a biquaternion provides us up to eight-dimensional formulations in Minkowski space.

3 Generalized Maxwell-Proca–type equations of gravity

In this section, we prefer to start with a short review of the gravitational theory in order to establish the notation of its Proca-type generalization and make the analogies with electromagnetic counterparts clear. The similarity between Newton's law of gravity and Coulomb's law of electricity leads to some formulations which are almost identical to the Maxwell equations of electromagnetism. In gravity, the source of a gravitational field is the mass of the body while the source of an electric field in electromagnetic theory is the charge of the particle. In a similar way, the moving matter (mass current) generates a gravitomagnetic field according to Einstein's general relativity just as a magnetic field is produced in Maxwell's equations by the moving charges (electric current). Hence, the set of governing equations related to gravity can be expressed in the similar form with their electromagnetic counterparts. This theoretical analogy between the Maxwell equations and the gravity was first proposed by Heaviside [24] in 1893.

The gravitoelectric (GE) $\tilde{\boldsymbol{E}}$ and gravitomagnetic (GM) $\tilde{\boldsymbol{H}}$ fields satisfy the following the Maxwell-type field equations for linear gravitation (with $G = c = 1$),

$$\nabla \cdot \tilde{\boldsymbol{E}} = -\varrho_e, \tag{9a}$$
$$\nabla \cdot \tilde{\boldsymbol{H}} = 0, \tag{9b}$$
$$\nabla \times \tilde{\boldsymbol{E}} = -\frac{\partial \tilde{\boldsymbol{H}}}{\partial t}, \tag{9c}$$
$$\nabla \times \tilde{\boldsymbol{H}} = -\tilde{\boldsymbol{J}}^e + \frac{\partial \tilde{\boldsymbol{E}}}{\partial t}, \tag{9d}$$

where ϱ_e and $\tilde{\boldsymbol{J}}^e$ are the gravitoelectric mass density and gravitoelectric mass current density, respectively [25]. The above-mentioned equations are termed as gravitoelectromagnetic equations. Similarly to electromagnetism, $\tilde{\boldsymbol{E}}$ and $\tilde{\boldsymbol{H}}$ fields are defined in terms of the scalar potential φ_e and vector potential $\tilde{\boldsymbol{A}}^e$ of gravitoelectromagnetism (GEM),

$$\tilde{\boldsymbol{E}} = -\nabla\varphi_e - \frac{\partial \tilde{\boldsymbol{A}}^e}{\partial t}, \tag{10a}$$
$$\tilde{\boldsymbol{H}} = \nabla \times \tilde{\boldsymbol{A}}^e. \tag{10b}$$

Similarly, the Lorenz gauge condition in GEM is provided by

$$\nabla \cdot \tilde{\boldsymbol{A}}^e + \frac{\partial \varrho_e}{\partial t} = 0, \tag{11}$$

which defines the law of mass conservation.

The Proca-type equations for GEM can be derived in the case of non-zero rest mass of gravitons ($m_g \neq 0$). Argyris and Ciubotariu [12] have defined a model of the universe (a gravitating system) as a collection of dust particles and non-zero rest mass gravitons which exhibits collective mode behaviour. In this model, the required Proca-type gravitational field equations have similar forms with their electromagnetic counterparts,

$$\nabla \cdot \tilde{\boldsymbol{E}} = -\varrho_e - \mu_\gamma^2\varphi_e, \tag{12a}$$
$$\nabla \cdot \tilde{\boldsymbol{H}} = 0, \tag{12b}$$
$$\nabla \times \tilde{\boldsymbol{E}} = -\frac{\partial \tilde{\boldsymbol{H}}}{\partial t}, \tag{12c}$$
$$\nabla \times \tilde{\boldsymbol{H}} = -\tilde{\boldsymbol{J}}^e + \frac{\partial \tilde{\boldsymbol{E}}}{\partial t} - \mu_\gamma^2\tilde{\boldsymbol{A}}^e. \tag{12d}$$

Here $\tilde{\boldsymbol{E}}$ and $\tilde{\boldsymbol{H}}$ are related to the potentials φ_e and $\tilde{\boldsymbol{A}}^e$ just as given in eq. (10). As defined for the Proca equation in electromagnetic theory, the term

$$\mu_\gamma = \frac{m_g c}{\hbar}, \tag{13}$$

is the inverse Compton wavelength of the graviton.

In the electromagnetic theory, assuming the existence of magnetic charges leads Maxwell's equations to show more symmetry between the electric and magnetic fields. In a similar way, according to the linearized Einstein's theory, the existence of GM monopoles can be proposed to be analogous to the magnetic charges. The GM monopole term is also termed by some authors as the GM charge or GM mass [26]. The GM monopole can be seen as the source of the GM field just as the mass (GE charge) is the source of the GE field. By assuming the existence of the GM monopole in

GEM theory, the Maxwell-like expressions in eq. (9) also need to be modified to get more symmetry between the GE and GM fields as [6]:

$$\nabla\cdot\tilde{\boldsymbol{E}} = -\varrho_e, \tag{14a}$$
$$\nabla\cdot\tilde{\boldsymbol{H}} = -\varrho_m, \tag{14b}$$
$$\nabla\times\tilde{\boldsymbol{E}} = \tilde{\boldsymbol{J}}^m - \frac{\partial\tilde{\boldsymbol{H}}}{\partial t}, \tag{14c}$$
$$\nabla\times\tilde{\boldsymbol{H}} = -\tilde{\boldsymbol{J}}^e + \frac{\partial\tilde{\boldsymbol{E}}}{\partial t}. \tag{14d}$$

Here, ϱ_m and $\tilde{\boldsymbol{J}}^m$ are the gravitomagnetic mass density and the gravitomagnetic mass current density, respectively. By taking the divergence of both sides of the third and the last equations,

$$\nabla\cdot(\nabla\times\tilde{\boldsymbol{E}}) = \nabla\cdot\tilde{\boldsymbol{J}}^m - \frac{\partial}{\partial t}(\nabla\cdot\tilde{\boldsymbol{H}}), \tag{15}$$
$$\nabla\cdot(\nabla\times\tilde{\boldsymbol{H}}) = -\nabla\cdot\tilde{\boldsymbol{J}}^e + \frac{\partial}{\partial t}(\nabla\cdot\tilde{\boldsymbol{E}}), \tag{16}$$

and using the first and the second definitions in eq. (14), we obtain the following continuity equations:

$$\nabla\cdot\tilde{\boldsymbol{J}}^m + \frac{\partial\varrho_m}{\partial t} = 0, \tag{17a}$$
$$\nabla\cdot\tilde{\boldsymbol{J}}^e + \frac{\partial\varrho_e}{\partial t} = 0. \tag{17b}$$

In addition to the mentioned potential in eq. (10), two new potentials, a vector potential $\tilde{\boldsymbol{A}}^m$ and a scalar potential φ_m, are introduced in order to symmetrize the Maxwell-like equations in gravity. Consequently, the expressions of $\tilde{\boldsymbol{E}}$ and $\tilde{\boldsymbol{H}}$ also need modification in the following approach:

$$\tilde{\boldsymbol{E}} = -\nabla\varphi_e - \frac{\partial\tilde{\boldsymbol{A}}^e}{\partial t} - \nabla\times\tilde{\boldsymbol{A}}^m, \tag{18a}$$
$$\tilde{\boldsymbol{H}} = -\nabla\varphi_m - \frac{\partial\tilde{\boldsymbol{A}}^m}{\partial t} + \nabla\times\tilde{\boldsymbol{A}}^e. \tag{18b}$$

Furthermore, one more Lorenz condition related to new potentials must be satisfied,

$$\nabla\cdot\tilde{\boldsymbol{A}}^m + \frac{\partial\varphi_m}{\partial t} = 0. \tag{19}$$

In the next stage, by introducing the Proca-type massive graviton terms φ_e and $\tilde{\boldsymbol{A}}^e$ into gravitational field equations in eq. (14) we arrive at the generalized Maxwell-like equations for the GEM in the vector algebra formalism

$$\nabla\cdot\tilde{\boldsymbol{E}} = -\varrho_e - \mu_\gamma^2\varphi_e, \tag{20a}$$
$$\nabla\cdot\tilde{\boldsymbol{H}} = -\varrho_m, \tag{20b}$$
$$\nabla\times\tilde{\boldsymbol{E}} = \tilde{\boldsymbol{J}}^m - \frac{\partial\tilde{\boldsymbol{H}}}{\partial t}, \tag{20c}$$
$$\nabla\times\tilde{\boldsymbol{H}} = -\tilde{\boldsymbol{J}}^e + \frac{\partial\tilde{\boldsymbol{E}}}{\partial t} - \mu_\gamma^2\tilde{\boldsymbol{A}}^e. \tag{20d}$$

These expressions are the most generalized forms of the Maxwell-like GEM equations as they include both gravitomagnetic monopole and Proca-type terms. In this section, we have discussed the possible existence of GEM fields in nature. In the next section, we aim to formulate the biquaternionic field equations and derive alternative formulations in a compact and elegant approach.

4 Biquaternionic form of Proca-type gravitoelectromagnetic equations with gravitomagnetic monopoles

In order to formulate biquaternionic Proca-type GEM equations, we will define the biquaternionic differential operator

$$\Box = i\frac{\partial}{\partial t} + \nabla = i\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\boldsymbol{e}_1 + \frac{\partial}{\partial y}\boldsymbol{e}_2 + \frac{\partial}{\partial z}\boldsymbol{e}_3, \tag{21}$$

and a biquaternion

$$\tilde{\mathbf{F}} = \tilde{\boldsymbol{E}} + i\tilde{\boldsymbol{H}} = (\tilde{E}_1 + i\tilde{H}_1)\boldsymbol{e}_1 + (\tilde{E}_2 + i\tilde{H}_2)\boldsymbol{e}_2 + (\tilde{E}_3 + i\tilde{H}_3)\boldsymbol{e}_3 \tag{22}$$

that combines the gravitoelectric and gravitomagnetic fields in a single expression. Here $\tilde{E}_1$, $\tilde{H}_1$, $\tilde{E}_2$, $\tilde{H}_2$, $\tilde{E}_3$, $\tilde{H}_3$ are the components of the $\tilde{\boldsymbol{E}}$ and $\tilde{\boldsymbol{H}}$ vector fields along the x-, y- and z-direction, respectively.

The operation of the biquaternionic differential operator on $\tilde{\mathbf{F}}$ yields

$$\Box\tilde{\mathbf{F}} = \left[i\frac{\partial}{\partial t} + \nabla\right][\tilde{\mathbf{H}} + i\tilde{\mathbf{E}}] = \left[-\nabla\cdot\tilde{\boldsymbol{H}} + \nabla\times\tilde{\boldsymbol{H}} - \frac{\partial\tilde{\boldsymbol{E}}}{\partial t}\right] + i\left[-\nabla\cdot\tilde{\boldsymbol{E}} + \nabla\times\tilde{\boldsymbol{E}} + \frac{\partial\tilde{\boldsymbol{H}}}{\partial t}\right].$$

By using the expressions in eq. (20), then we arrive at the following expression:

$$\Box\tilde{\mathbf{F}} = [\varrho_m + i\varrho_e] + \left[-\tilde{\boldsymbol{J}}^e + i\tilde{\boldsymbol{J}}^m\right] - \mu_\gamma^2\left[-i\varphi_e + \tilde{\boldsymbol{A}}^e\right]. \tag{23}$$

We can introduce a new biquaternion that we termed as biquaternionic generalized gravitational source density

$$\tilde{\mathbf{J}} = \varrho + \tilde{\boldsymbol{J}} = (\varrho_m + i\varrho_e)\boldsymbol{e}_0 + (-\tilde{J}_1^e + i\tilde{J}_1^m)\boldsymbol{e}_1 + (-\tilde{J}_2^e + i\tilde{J}_2^m)\boldsymbol{e}_2 + (-\tilde{J}_3^e + i\tilde{J}_3^m)\boldsymbol{e}_3, \tag{24}$$

where

$$\varrho = \varrho_m + i\varrho_e \tag{25}$$

is the generalized gravitational mass density, and

$$\tilde{\boldsymbol{J}} = (-\tilde{J}_1^e + i\tilde{J}_1^m)\boldsymbol{e}_1 + (-\tilde{J}_2^e + i\tilde{J}_2^m)\boldsymbol{e}_2 + (-\tilde{J}_3^e + i\tilde{J}_3^m)\boldsymbol{e}_3 \tag{26}$$

is the generalized gravitational current source density. Similarly, by defining the biquaternionic GE potential as

$$\tilde{\mathbf{A}}^e = -i\varphi_e + \tilde{\boldsymbol{A}}^e, \tag{27}$$

eq. (23) can be transcribed in the following form:

$$\Box\tilde{\mathbf{F}} = \varrho + \tilde{\boldsymbol{J}} - \mu_\gamma^2\tilde{\mathbf{A}}^e. \tag{28}$$

According to the definitions in eq. (24), all of the gravitational field equations provided in eq. (20) can be expressed as the following compact form:

$$\Box\tilde{\mathbf{F}} + \mu_\gamma^2\tilde{\mathbf{A}}^e = \tilde{\mathbf{J}}. \tag{29}$$

This biquaternionic equation is the most generalized formulation of the Proca-type GEM equations with monopole terms. Thus, all Maxwell-Proca-Heaviside–type expressions of the GEM are rewritten in a single equation.

We can define the gravitational generalized potential in biquaternion form,

$$\tilde{\mathbf{A}} = \varphi + \tilde{\boldsymbol{A}} = -(\varphi_m + i\varphi_e) + (\tilde{\boldsymbol{A}}^e - i\tilde{\boldsymbol{A}}^m), \tag{30}$$

where

$$\varphi = -(\varphi_m + i\varphi_e) \tag{31}$$

is the generalized gravitational scalar potential, and

$$\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}^e - i\tilde{\boldsymbol{A}}^m = (\tilde{A}_1^e - i\tilde{A}_1^m)\boldsymbol{e}_1 + (\tilde{A}_2^e - i\tilde{A}_2^m)\boldsymbol{e}_2 + (\tilde{A}_3^e - i\tilde{A}_3^m)\boldsymbol{e}_3 \tag{32}$$

is the generalized gravitational vector potential. Operating the conjugate of the differential operator $\Box^*$ on $\tilde{\mathbf{A}}$

$$\Box^*\tilde{\mathbf{A}} = \left[-i\frac{\partial}{\partial t} + \nabla\right]\left[(-\varphi_m + \tilde{\boldsymbol{A}}^e) + i(-\varphi_e - \tilde{\boldsymbol{A}}^m)\right],$$

yields

$$\Box^*\tilde{\mathbf{A}} = \left[-\frac{\partial\varphi_e}{\partial t} - \nabla\cdot\tilde{\boldsymbol{A}}^e + \nabla\times\tilde{\boldsymbol{A}}^e - \nabla\varphi_m - \frac{\partial\tilde{\boldsymbol{A}}^m}{\partial t}\right] + i\left[\frac{\partial\varphi_m}{\partial t} + \nabla\cdot\tilde{\boldsymbol{A}}^m - \nabla\times\tilde{\boldsymbol{A}}^m - \nabla\phi_e - \frac{\partial\tilde{\boldsymbol{A}}^e}{\partial t}\right]. \tag{33}$$

According to the definitions in eq. (18) with two Lorenz gauge conditions in eqs. (11) and (19), the vectorial parts in the first and the second parenthesis yield the fields $\tilde{\boldsymbol{H}}$ and $\tilde{\boldsymbol{E}}$, respectively. In the light of eq. (22), we obtain

$$\Box^*\tilde{\mathbf{A}} = \tilde{\mathbf{H}} + i\tilde{\mathbf{E}} = \tilde{\mathbf{F}}. \tag{34}$$

The re-operation of the biquaternionic differential operator $\Box$ on this equation yields

$$\Box\Box^*\tilde{\mathbf{A}} = \Box\tilde{\mathbf{F}}. \tag{35}$$

By using the compact expression derived in eq. (29), we obtain the following equation:

$$\boxdot\tilde{\mathbf{A}} = \tilde{\mathbf{J}} - \mu_\gamma^2\tilde{\mathbf{A}}^e. \tag{36}$$

Here $\boxdot$ is the d'Alembertian operator which is provided by

$$\boxdot = \Box\Box^* = \Box^*\Box = \left[i\frac{\partial}{\partial t} + \nabla\right]\left[-i\frac{\partial}{\partial t} + \nabla\right] = \left[\frac{\partial^2}{\partial t^2} - \nabla^2\right]. \tag{37}$$

Collecting the similar terms on one side, finally we arrive at the gravitational wave equation with the Proca-type generalization

$$\boxdot\tilde{\mathbf{A}} + \mu_\gamma^2\tilde{\mathbf{A}}^e = \tilde{\mathbf{J}}. \tag{38}$$

Actually, this expression is the most general form of gravitational wave equation as it includes both massive graviton and gravitomagnetic monopole terms. If eq. (38) is clearly expressed as

$$\left[\frac{\partial^2}{\partial t^2} - \nabla^2\right]\left[-(\varphi_m + i\varphi_e) + (\tilde{\boldsymbol{A}}^e - i\tilde{\boldsymbol{A}}^m)\right] + \mu_\gamma^2\left[-i\varphi_e + \tilde{\boldsymbol{A}}^e\right] = [\varrho_m + i\varrho_e] + \left[-\tilde{\boldsymbol{J}}^e + i\tilde{\boldsymbol{J}}^m\right], \tag{39}$$

and separated into its real and imaginary parts with scalar and vector terms, we arrive at the following expressions:

$$\frac{\partial^2\varphi_e}{\partial t^2} - \nabla^2\varphi_e + \mu_\gamma^2\varphi_e = -\varrho_e, \tag{40a}$$

$$\frac{\partial^2\varphi_m}{\partial t^2} - \nabla^2\varphi_m = -\varrho_m, \tag{40b}$$

$$\frac{\partial^2\tilde{\boldsymbol{A}}^m}{\partial t^2} - \nabla^2\tilde{\boldsymbol{A}}^m = -\tilde{\boldsymbol{J}}_m, \tag{40c}$$

$$\frac{\partial^2\tilde{\boldsymbol{A}}^e}{\partial t^2} - \nabla^2\tilde{\boldsymbol{A}}^e + \mu_\gamma^2\tilde{\boldsymbol{A}}^e = -\tilde{\boldsymbol{J}}_e. \tag{40d}$$

In the absence of the GM monopole, the generalized Proca-type GEM equations in eq. (20) reduce to the expressions in eq. (12), then the ϱ_m, φ_m, $\tilde{\boldsymbol{J}}_m$, $\tilde{\boldsymbol{A}}_m$ terms in eqs. (24) and (30) vanish. Consequently, eq. (38) should be rewritten in the following manner:

$$\left[\boxdot + \mu_\gamma^2\right]\tilde{\mathbf{A}}^e = \tilde{\mathbf{J}}^e, \tag{41}$$

where $\tilde{\mathbf{J}}^e$ is the GE density and is defined as

$$\tilde{\mathbf{J}}^e = -\varrho_e + i\tilde{\boldsymbol{J}}^e. \tag{42}$$

Similarly to electromagnetism in a source-free region, the GE current density also vanishes, $\tilde{\mathbf{J}}^e = 0$, and eq. (41) reduces to

$$\left[\boxdot + \mu_\gamma^2\right]\tilde{\mathbf{A}}^e = 0, \tag{43}$$

which is essentially the biquaternionic Klein-Gordon equation for the graviton. We can also write this equation for the field variables

$$\frac{\partial^2\varphi_e}{\partial t^2} - \nabla^2\varphi_e + \mu_\gamma^2\varphi_e = 0, \tag{44a}$$

$$\frac{\partial^2\tilde{\boldsymbol{A}}^e}{\partial t^2} - \nabla^2\tilde{\boldsymbol{A}}^e + \mu_\gamma^2\tilde{\boldsymbol{A}}^e = 0. \tag{44b}$$

Consequently, we can state that the biquaternionic formalism proposed in this paper provides successful results for all types of extended form of GEM.

5 Conclusions

In this paper, we have proposed an alternative formulation based on biquaternions for all types of generalized gravity. Although Argyris and Ciubotariu [12] have proposed a Proca-type generalization of gravity based on non-zero rest mass of gravitons and obtained a result analogous to the massive electromagnetism, in the relevant papers, the biquaternionic formulation of this generalization is absent. Therefore, this paper fills a gap which was not formulated in similar studies before. We have combined the gravitomagnetic monopole terms with the Proca-type generalization of gravity and expressed them in an elegant, economical and compact form.

The proposed biquaternionic formalism provides similar results for the unified Maxwell equations with non-vanishing photon mass and Dirac monopoles. In this formalism, we have shown, in eq. (29), that all field equations of gravity can be transcribed as one biquaternionic equation. Moreover, in eq. (38), the gravitational wave equation with the Proca-type generalization and Heaviside monopoles has been obtained in a compact and elegant manner. Similarly, the most generalized form of the homogeneous Klein-Gordon equation has been developed in eq. (43) for the graviton. Consequently, the biquaternionic formalism used in this paper presents a compact, simpler and elegant tool for deriving the alternative formulations related to gravitoelectromagnetism.

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