Eur. Phys. J. H
DOI: 10.1140/epjh/e2012-30061-5

THE EUROPEAN PHYSICAL JOURNAL H

Fritz Hasenöhrl and E = mc^2

Stephen Boughn[a]

Haverford College, 19041 Haverford, PA, USA

Received 28 November 2012 / Received in final form 17 December 2012
Published online 21 January 2013

Abstract. In 1904, the year before Einstein's seminal papers on special relativity, Austrian physicist Fritz Hasenöhrl examined the properties of blackbody radiation in a moving cavity. He calculated the work necessary to keep the cavity moving at a constant velocity as it fills with radiation and concluded that the radiation energy has associated with it an apparent mass such that $E = \frac{3}{8}mc^2$. In a subsequent paper, also in 1904, Hasenöhrl achieved the same result by computing the force necessary to accelerate a cavity already filled with radiation. In early 1905, he corrected the latter result to $E = \frac{3}{4}mc^2$. This result, i.e., $m = \frac{4}{3}E/c^2$, has led many to conclude that Hasenöhrl fell victim to the same "mistake" made by others who derived this relation between the mass and electrostatic energy of the electron. Some have attributed the mistake to the neglect of stress in the blackbody cavity. In this paper, Hasenöhrl's papers are examined from a modern, relativistic point of view in an attempt to understand where he went wrong. The primary mistake in his first paper was, ironically, that he didn't account for the loss of mass of the blackbody end caps as they radiate energy into the cavity. However, even taking this into account one concludes that blackbody radiation has a mass equivalent of $m = \frac{4}{3}E/c^2$ or $m = \frac{5}{3}E/c^2$ depending on whether one equates the momentum or kinetic energy of radiation to the momentum or kinetic energy of an equivalent mass. In his second and third papers that deal with an accelerated cavity, Hasenöhrl concluded that the mass associated with blackbody radiation is $m = \frac{4}{3}E/c^2$, a result which, within the restricted context of Hasenöhrl's gedanken experiment, is actually consistent with special relativity. (If one includes all components of the system, including cavity stresses, then the total mass and energy of the system are, to be sure, related by $m = E/c^2$.) Both of these problems are non-trivial and the surprising results, indeed, turn out to be relevant to the "$\frac{4}{3}$ problem" in classical models of the electron. An important lesson of these analyses is that $E = mc^2$, while extremely useful, is not a "law of physics" in the sense that it ought not be applied indiscriminately to any extended system and, in particular, to the subsystems from which they are comprised. We suspect that similar problems have plagued attempts to model the classical electron.

[a] e-mail: sboughn@haverford.edu

1 Historical introduction

In 1904-5 Fritz Hasenöhrl published the three papers, all with the title "On the theory of radiation in moving bodies," for which he is best known ([Hasenöhrl 1904a; Hasenöhrl 1904b; Hasenöhrl 1905] referred to as H1, H2 and H3). They concerned the mass equivalent of blackbody radiation in a moving cavity. The latter two papers appeared in the *Annalen der Physik* and for his work Hasenöhrl won the Haitinger Prize of the Austrian Academy of Sciences. (In 1907 he succeeded Boltzmann as professor of theoretical physics at the University of Vienna.) These three papers analyzed two different *gedankenexperiments* each of which demonstrated a connection between the energy of radiation and inertial mass. In the first thought experiment, he arrived at $E = \frac{3}{8}mc^2$ and in the second, $E = \frac{3}{4}mc^2$. Hasenöhrl was working within the confines of an ether theory and, not surprisingly, these results were soon replaced by Einstein's quintessential $E = mc^2$. Even so, it is interesting to ask "Where did Hasenöhrl go wrong?"

The notion that mass and energy are related originated well before Hasenöhrl's and Einstein's papers. As early as 1881, J.J. Thomson [Thomson 1881] argued that the backreaction of the field of a charged sphere (the classical model of the electron) would impede its motion and result in an apparent mass increase of $(4/15)\mu e^2/a$, where e was the charge on the sphere, a its radius and μ the magnetic permeability. Fitzgerald, Heaviside, Wein, and Abraham subsequently corrected Thompson's analysis and all concluded that the interaction of a moving electron with its field results in an apparent mass given by $m = \frac{4}{3}E/c^2$ where E is the electrostatic energy of the stationary electron. (For more on these early works, see Max Jammer's *Concepts of Mass* [Jammer 1951].)

All these investigations were of the relationship between the mass and the electrostatic energy of the electron. Hasenöhrl broadened the query by asking "what is the mass equivalent of blackbody radiation?" Previous explanations as to why Hasenöhrl failed to achieve the "correct" result, i.e., $m = E/c^2$, are not particularly illuminating. For example, in his *Concepts of Mass in Contemporary Physics and Philosophy* [Jammer 2000], Jammer, says only: "What was probably the most publicized prerelativistic declaration of such a relationship [between inertia and energy] was made in 1904 by Fritz Hasenöhrl. Using Abraham's theory, Hasenöhrl showed that a cavity with perfectly reflecting walls behaves, if set in motion, as if it has a mass m given by $m = 8V\varepsilon_0/3c^2$, where V is the volume of the cavity, ε_0 is the energy density at rest, and c is the velocity of light." (For a more extensive discussion, see Boughn & Rothman 2011.) The overall impression is that few authors have made an effort to understand exactly what Hasenöhrl did.

In certain ways Hasenöhrl's thought experiments were both more bold and more well defined than Einstein's, which alone renders them worthy of study. A macroscopic, extended cavity filled with blackbody radiation is certainly a more complicated system than Einstein's point particle emitting back-to-back photons. In addition, whereas the characteristics of blackbody radiation and the laws governing the radiation (Maxwell's equations) were well known at the time, the emission process of radiation from point particles (atoms) was not well understood. Einstein simply conjectured that the details of the emission process were not relevant to his result.

Another reason to investigate Hasenöhrl's thought experiments is the apparent relation to the famous "$\frac{4}{3}$ problem" of the self-energy of the electron (see Section 4). Enrico Fermi, in fact, assumed that the two $\frac{4}{3}$'s were identical and devoted one of his earliest papers to resolving the issue [Fermi & Pontremoli 1923b]. We initially attempted to understand Hasenöhrl's apparently incorrect results by reproducing his analyses. This effort was frustrated by his cumbersome, pre-relativistic calculations that were not free from error. The objective of the present paper is to introduce

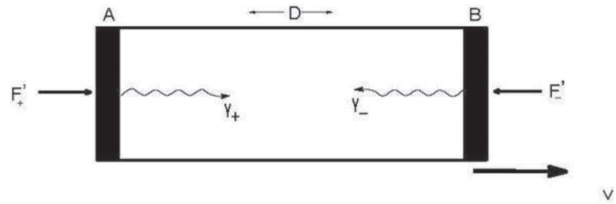

Fig. 1. A cavity consisting of two blackbody radiators, A and B in a completely reflecting enclosure of length D. At a time $t = 0$ the radiating caps suddenly begin to emit radiation in the direction of motion (+) and opposite the direction of motion (–). From the frame of a moving observer, the (+/–) radiation will be blue/red shifted and hence exert different reaction forces on A and B. (Based on H1, H2.)

Hasenöhrl's two thought experiments and then achieve correct relativistic results (Sections 2 and 3), which will allow us to understand both the limtations and strengths of his proofs. In the process we determine that the neglect of cavity stresses is not the primary issue and that Fermi's proof is apparently violated by Hasenöhrl's gedanken experiment (Section 4).

2 Hasenöhrl's first thought experiment

Considering the importance of blackbody radiation at the turn of the 20th century, an investigation of the properties of blackbody radiation in a moving cavity was an eminently reasonable undertaking. Hasenöhrl considered the case of two blackbody radiators (endcaps) at temperature T enclosed in a cylindrical cavity made of reflecting walls (see Figure 1). Initially the cavity is assumed to be void of any radiation and at a time $t = 0$ the two radiators A and B are, in some unspecified way, enabled to begin filling the cavity with radiation. He assumes that the blackbody radiators have sufficiently large heat capacity that they do not cool appreciably during this process. The two endcaps are presummably held in place by stresses in the cavity walls; although, Hasenöhrl refers to these forces as external (*außen*) and treats them as such. Whether he actually viewed the radiators as being held in place by forces external to the cavity or by internal stresses makes no difference to his subsequent analysis. We choose to make this explicit by supposing that the two encaps are actually held in place by external forces and are otherwise free to slide back and forth inside the cavity.

In the rest frame of the cavity, the radiation reaction forces on the two endcaps are equal and opposite as are the external forces required to hold the endcaps in place. As viewed by a moving observer, however, the situation is quite different. In this observer's frame, the radiation from the trailing endcap (A) is Doppler shifted to the blue while radiation from the leading endcap (B) is redshifted. Therefore, when the radiators are switched on, the moving observer finds that two different external forces F_+ and F_- are required to counter the radiation reaction forces on the endcaps and keep them moving at a constant velocity. Because the endcaps are in motion and because $F_+ \neq F_-$, the interesting consequence is that net work is performed on the cavity.

Hasenöhrl does not use the terminology "rest frame of the cavity" or "lab frame." Although he mentions the *ether* only three times throughout his papers, it is clear that for him all motion is taking place relative to the absolute frame of the ether. In this paper the lab and cavity frames have their usual meanings. Quantities referring to the lab frame are designated by a prime; cavity-frame quantities are unprimed.

The crux of Hasenöhrl's analysis is a calculation of the work done by the external forces from the time that the blackbody radiators are turned on to the time that the cavity is at equilibrium and filled with blackbody radiation. To order v^2/c^2, it turns out that Hasenöhrl's result, $W' = \frac{4}{3}E(v^2/c^2)$, is precisely the same as given by special relativity (to the same order in v/c). For this reason and because Hasenöhrl's pre-relativistic calculation is very difficult to follow, we use a proper relativistic analysis to compute the value of the radiation reaction forces and the work performed by the external forces required to balance them. That the two results agree is not surprising because the reaction forces are only needed to first order in v/c and can be derived from the non-relativistic Doppler shift and abberation relations.

2.1 Relativistic calculation of the work

The strategy is to calculate the radiation pressure on a moving surface by transforming the blackbody radiation intensity i in the cavity rest frame to a frame moving at velocity v relative to the cavity. A detailed derivation of this tranformation can be found in Boughn & Rothman [2011]; however, the same result can be got more directly from an expression for the anisotropic temperature of the cosmic background radiation [Peebles & Wilkinson 1968]. Peebles and Wilkinson found that in a moving frame the radiation maintains a blackbody spectrum

$$i'_\nu = \frac{2h\nu'^3}{c^2}(e^{h\nu'/kT'-1})^{-1} \tag{2.1}$$

but with a temperature T' that depends on direction,

$$T(\theta') = T(\frac{(1-\beta^2)^{\frac{1}{2}}}{(1-\beta\cos\theta')} \tag{2.2}$$

where T is the blackbody temperature in the rest (cavity) frame, $\beta \equiv v/c$, and θ' the angle between the radiation and $\mathbf{v}$. The integral of i'_ν over all frequencies is well known to be $\propto T'^4$. Therefore the intensity in the lab frame is

$$i' = i\frac{(1-\beta^2)^2}{(1-\beta\cos\theta')^4} \tag{2.3}$$

where i is the intensity in the cavity frame and is given by the usual Planck formula. Although this derivation of Eq. (2.3) assumes blackbody radiation, it is straightforward to show that it holds for any isotropic radiation field [Boughn & Rothman 2011]. It will become apparent that in order to calculate the work W' to second order in β, one can ignore terms of order β^2 in Eq. (2.3), i.e., the relativistic corrections. That one need only use the non-relativistic transformation laws to compute the work at this order provides an explanation as to why Hasenöhrl obtained an essentially correct result for the work. Finally, it is well known that the intensity and energy density of blackbody radiation (or any isotropic radiation field for that matter) is

$$\rho = \frac{4\pi i}{c}, \tag{2.4}$$

which Hasenöhrl accepts (H1).

Now consider the radiation being emitted at an angle θ' from the left end cap of the cavity in Figure 1. Using the relation between momentum and energy for electromagnetic radiation $P = E/c$, the rate at which momentum leaves that end cap is given by

$$\frac{dP'}{dt'} = \frac{i'}{c^2}\, d\Omega' A\, (c\cos\theta' - v), \tag{2.5}$$

where A is the area of end cap. By symmetry, the only non-vanishing component of the momentum is in the direction of v. The last factor in Eq. (2.5) is due to the lab frame relative velocity between the radiation and the encap in this direction. From Newton's third law, dP'/dt' is the magnitude of the rightward external force needed to counter the radiation reaction force and keep the left endcap moving at constant velocity. The work done by that portion of the external force needed to counter the reaction force of the radiation emitted at angle θ' is

$$dW'_+(\theta') = \frac{dP'}{dt'} v\Delta t'(\theta'), \tag{2.6}$$

where $\Delta t'(\theta')$ is the light-crossing time for radiation at an angle θ'. After this time, radiation will be absorbed by end cap B and the force necessary to counter the resulting radiation pressure on B is equal and opposite to the force on end cap A. It is straightforward to show that, independent of the number of reflections along the cavity side wall,

$$\Delta t'(\theta') = \frac{D'}{c\cos\theta' - v)} \tag{2.7}$$

where $D' = D/\gamma$ is the Lorentz contracted length of the cavity in the lab frame. For cylindrical symmetry $d\Omega' = 2\pi \sin\theta' d\theta' = -2\pi d(\cos\theta')$ and hence

$$dW'_+(\theta') = -2\pi i \frac{AD'v(1-\beta^2)\cos\theta' d(\cos\theta')}{c^2(1-\beta\cos\theta')^4}. \tag{2.8}$$

Retaining terms to first order in β, the total work on radiator A is

$$\begin{aligned} W'_+ &= \frac{2\pi i ADv}{c^2}\int_0^1 (1+4\beta\cos\theta')\cos\theta' d(cos\theta') \\ &= \frac{2\pi i ADv}{c^2}\left(\frac{1}{2}+\frac{4\beta}{3}\right). \end{aligned} \tag{2.9}$$

Note that, because of abberation, the upper limit on the above integral is not precisely unity; however, this correction is also higher order in β and can be neglected. The work on the right end cap can be found by taking the negative of the above expression after reversing the sign on β. The net work done by the external force is consequently

$$W' = W'_+ + W'_- = \frac{4}{3}\left[\frac{4\pi i AD}{c}\right]\frac{v^2}{c^2} \tag{2.10}$$

where $AD = V$ is the rest frame volume of the cavity. From Eq. (2.4), the quantity in brackets is $\rho V = E$, the energy of the blackbody radiation in the cavity rest frame. Therefore,

$$W' = \frac{4}{3}E\frac{v^2}{c^2}, \tag{2.11}$$

which is exactly Hasenöhrl's result.

One might worry that that we have ignored questions of simultaneity that, afterall, are first order in v/c. If the two endcaps begin radiating at the same time in the cavity rest frame, then in the moving lab frame, to first order in v/c, the trailing endcap will begin radiating $\delta t' = vD/c^2$ earlier. However, the time interval $\Delta t'$ (see Eq. (2.7)) used to compute the work is the lab frame time interval required for radiation emitted from endcap A to reach endcap B, a quantity that is independent of when the radiation is emitted from A. The same is true for the radiation emitted from endcap B and absorbed by endcap A. The only difficulty that might arise is if encap B were required to absorb radiation before it begins to emit radiation (and the same situation for endcap A). However, the shortest length of time rquired for this to happen is D/c, the light travel time across the cavity, and this is much greater than $\delta t'$ from above.

Even so, Hasenöhrl's calculation is not without error. As pointed out in the abstract, his primary mistake was the ironic omission of the mass loss of the end caps as they radiate energy into the cavity. Newton's second law implies that an external force must be applied to an object that is loosing mass if that mass is to maintain a constant velocity. To first order in v/c, it is sufficient to consider the non-relativistic expression of the second law, i.e.,

$$F' = dP'/dt' = d(mv)/dt' = vdm/dt' + mdv/dt' = vdm/dt', \tag{2.12}$$

where $F' = F'_{ext} + F'_{rad}$, F'_{ext} is the external force, and F'_{rad} is the reaction force of the radiation on the end caps. Thus

$$F'_{ext} = vdm/dt' - F'_{rad} \tag{2.13}$$

and the work due to the external force is just

$$W' = \int F'_{ext} v dt' = \int v^2 dm - \int v F'_{rad} dt'. \tag{2.14}$$

We have already computed the second term on the right. From Eq. (2.11), it is just Hasenöhrl's $\frac{4}{3}E\beta^2$. The first term on the right is simply $\Delta m v^2$. It is now necessary to use the relativistic result that Δm must be equal to minus the energy lost by the end caps (divided by c^2), i.e., $\Delta m = -E/c^2$ where E is the energy radiated into, and therefore the energy content of, the cavity. Thus, the total work performed by external forces is

$$W' = -E\frac{v^2}{c^2} + \frac{4}{3}E\frac{v^2}{c^2} = \frac{1}{3}E\frac{v^2}{c^2}. \tag{2.15}$$

Whereas Hasenöhrl equated the external work to the kinetic energy of the radiation in the cavity, we must now consider the entire energy of the system, radiation plus blackbody end caps A and B. We again use the relativistic result that the change of energy of the end caps is, in the lab frame, given by $\gamma \Delta m c^2$ which to second order in β, is given by $-E(1 + 1/2\beta^2)$.[1] Then conservation of energy yields

$$W' = \frac{1}{3}E\beta^2 = E' - E\left(1 + \frac{1}{2}\beta^2\right). \tag{2.16}$$

Finally, we define the kinetic energy of the radiation to be

$$(E' - E) = \frac{5}{6}E\beta^2. \tag{2.17}$$

[1] One might argue that it is inappropriate to use the relativistic results $\Delta m = -E/c^2$ and $E' = \gamma \Delta m c^2$ in an analysis that purports to derive mass-energy equivalence. However, the reader is reminded that the present anlaysis is, indeed, relativistic and these two relations are known to be true for any bound, stable system by virture of the theorems of von Laue [1911] and Klein [1918] (see Section 4).

If one were to interpret this kinetic energy as due to an effective mass of the radiation, as did Hasenöhrl, i.e., $(E' - E) = \frac{1}{2} m_{eff} v^2$, then one finds that

$$m_{eff} = \frac{5}{3} E/c^2, \tag{2.18}$$

This value is not Hasenöhrl's $\frac{8}{3} E/c^2$; however, neither is it E/c^2 as one might expect from special relativity.

One might also choose to determine the effective mass from momentum conservation rather than energy conservation. Because the velocity is constant, this result is easily deduced from the above analysis. The total momentum impulse to the system delivered by external forces is

$$\Delta P'_{ext} = \int F'_{ext} dt' = \frac{1}{v} \int v F'_{ext} dt' = W'/v = \frac{1}{3} E \frac{v}{c^2}. \tag{2.19}$$

The change in momentum of the end caps, to first order in v/c, is $\Delta m v = -Ev/c^2$. Therefore, by conservation of momentum,

$$\Delta P'_{ext} = \Delta m v + P'_{rad} = -(E/c^2)v + P'_{rad} \tag{2.20}$$

where P'_{rad} is the net momentum of the radiation. Then from Eq. (2.19)

$$P'_{rad} = \frac{4}{3} (E/c^2) v. \tag{2.21}$$

Attributing this momentum to an effective mass of the radiation, i.e., $P'_{rad} = m_{eff} v$, implies that

$$m_{eff} = \frac{4}{3} E/c^2, \tag{2.22}$$

which is different from both of the results discussed above. In order to make sense of all this, we turn to the special relativistic definition of energy and momentum for radiation.

2.2 Energy-momentum tensor

It is straighforward to calculate a lab frame expression for the radiative energy in the cavity using Eq. (2.3) and integrating over the times it takes radiation from the two end caps to fill the cavity (see Eq. 2.7). The total radiative momentum in the lab frame can be computed in the same way, noting that radiative momentum is radiative energy divided by the speed of light and taking into account the opposite directions of the momenta emitted from the two end caps. A more direct way of obtaining these results is simply by transforming the energy-momentum tensor of the the radiation from the cavity frame to the lab frame. In addition, this formalism will be useful in our anlaysis of Hasenöhrl's second gedanken experiment in Section 3.1 below.

The energy-momentum tensor for blackbody radiation (in the cavity frame) is the same as for a perfect fluid with equation of state $p = \rho/3$ and has the form, i.e., $T^{00} = \rho$, $T^{0i} = T^{i0} = 0$, $T^{ij} = p\,\delta_{ij}$, where ρ and p represent the energy density and pressure of the radiation in the cavity frame. Because $T^{\mu\nu}$ is a tensor quantity, it is straightforward to express it in any frame as

$$T^{\mu\nu} = \frac{1}{c^2} (\rho + p) u^\mu u^\nu + \eta^{\mu\nu} p. \tag{2.23}$$

Here, all the symbols have their usual meanings: $\mathbf{u} \equiv (\gamma \mathbf{c}, \gamma \mathbf{v})$ is the four velocity of the frame, $\mathbf{v}$ is the three-velocity, $\gamma \equiv (1-\beta^2)^{-1/2}$ and the metric tensor $\eta^{\mu\nu} \equiv (-1,+1,+1,+1)$. Greek indices range from 0 to 3 and Latin indices take on the values 1 to 3. Thus, in the lab frame

$$T'^{00} = (\rho + p)\gamma^2 - p = \rho\gamma^2 + \frac{\rho}{3}(\gamma^2 - 1), \tag{2.24}$$

and

$$T'^{0x} = (\rho + p)\gamma^2 \frac{v}{c} = \frac{4}{3}\rho\gamma^2 \frac{v}{c} \tag{2.25}$$

where x indicates the direction of motion which is parallel to the cavity axis.

Because T'^{00} represents energy density, the total energy in the lab frame is

$$E' = \int T'^{00} dV' \tag{2.26}$$

where $dV' = V/\gamma$ is the volume element in the lab frame. Therefore,

$$E' = \gamma^{-1} T'^{00} V \tag{2.27}$$

and from Eq. (2.24),

$$E' = \gamma E \left(1 + \frac{\beta^2}{3}\right) = E\left(1 + \frac{5}{6}\beta^2\right) + \mathcal{O}(\beta^4). \tag{2.28}$$

This expression is the same as Eq. (2.17) and indicates that, to second order in β, the work W' in Eq. (2.15) is consistent with the relativistic expression for energy in Eq. (2.26).

Similarly, from Eq. (2.25), the total momentum of the radiation in the lab frame is

$$P' = \frac{1}{c}\int T'^{0x} dV', \tag{2.29}$$

or

$$P' = \frac{4}{3}E\gamma\frac{v}{c^2} = \frac{4}{3}E\frac{v}{c^2} + \mathcal{O}(\beta^3). \tag{2.30}$$

Likewise, that this expression is the same as Eq. (2.21) indicates that Eq. (2.29) is, indeed, the relativistic momentum of the blackbody radiation in the lab frame.

We are left with the dilemma that there seem to be two different effective masses, $m_{\mathit{eff}} = \frac{5}{3}E/c^2$ and $m_{\mathit{eff}} = \frac{4}{3}E/c^2$, associated with blackbody radiation and neither of these is the expected $m_{\mathit{eff}} = E/c^2$. This is a direct consequence that our definition of the total radiative energy and momentum, $\int T'^{\mu 0} dV'$, is not a covariant expression, i.e., (E', P'^i) is not a proper 4-vector. That the total energy/momentum of an extended system behaves this way lies at the center of the previously mentioned "$\frac{4}{3}$ problem" of the self-energy of the electron. We will return to this issue after analyzing Hasenöhrl's second gedanken experiment.

3 The slowly accelerating cavity

3.1 Hasenöhrl's second thought experiment

Hasenöhrl's first gedanken experiment, suddenly switching on two blackbody end-caps that subsequently fill a cavity with radiation, may perhaps seem a bit contrived.

A more natural process would be to accelerate a cavity already filled with blackbody radiation and this is precisely what Hasenöhrl considered in his second paper (H2). On the other hand, an accelerating blackbody cavity is a more complicated system. In particular, one must worry whether or not the radiation remains in thermal equilibrium during the acceleration and whether or not the accelerated blackbody endcaps change their emission properties. Hasenöhrl was well aware of such problems. He sought to mitigate them by imagining that the process be carried out reversibly/adiabatically by requiring the that the velocity change happens "infinitely slowly". He also envisioned blackbody endcaps with heat capacities so small that their heat contents were negligible; their only purpose is to thermalize the radiation. In our analysis, we obviate the problem of thermal equilibrium by assuming the acceleration has been in effect for a very long time so that the cavity comes to eqilibrium. Because of the absolute frame of the ether, this assumption wasn't availible to Hasenöhrl. Even so, in our analysis we must assume that the acceleration is small in the sense that $aD/c^2 \ll 1$.

As in his first gedanken experiment, Hasenöhrl computed the work required, in this case, to accelerate the cavity to a speed v. Initially, he obtained the same result as in H1, i.e., $W = \frac{4}{3}E\beta^2$, which implied that $m = \frac{8}{3}E/c^2$. After Abraham pointed out a simpler way to calculate the mass, as the derivative of the electromagnetic momentum with respect to velocity: $m = d(\frac{4}{3}Ev/c^2)/dv = \frac{4}{3}E/c^2$, Hasenöhrl uncovered a factor of two error in H2, which brought him into agreement with Abraham. He subsequently published the correction in paper H3. This is, perhaps, why some have concluded that Hasenöhrl did nothing different from Abraham. However, Abraham's analysis was of the classical electron, while Hasenöhrl's was of blackbody radiation.

Hasenöhrl's calculation in H2 is extremely involved. He did not calculate the work directly, but rather calculated the small change in energy of the already filled cavity due to an incremental change in velocity. He equated the difference between this energy and that radiated by the endcaps to the incremental work performed on the system. We now present a modern analysis of this gedanken experiment.

Suppose that the cavity is already filled with blackbody radiation and assume that the acceleration has been applied for a sufficiently long time that the cavity is in equilibrium. This doesn't violate the condition that the cavity is intially at rest in the lab frame; we simply choose the lab frame to be the inertial frame that is instantaneously comoving with the cavity at $t = 0$. We also assume that the blackbody end caps each radiate according to Planck's law when observed in an instantaneously co-moving inertial frame. That is, we assume that an *ideal* blackbody is not affected by acceleration. This is analogous to the special relativistic assumption that *ideal* clocks are not affected by acceleration. Of course, whether or not real blackbodies or real clocks behave this way is open to question; however, one might expect that this is the case for very small accelerations. In any case, this is our *ansatz* that we will justify later. Finally, we ignore the mass of the cavity. One needn't assume the mass is negligible but rather only that including it doesn't change the results of the analysis. This will be justified shortly using the results of Section 3.2.

With these assumptions, it is straightforward to demonstrate that, in an instaneously comoving frame, the radiation is isotropic at every point in the cavity. This follows directly from Liouville's theorem, i.e., phase space density is constant along every particle trajectory. For photons, phase space density is proportional to i_ν/ν^3 (e.g. Misner et al. 1973). We assume that at the blackbody end cap, i_ν is given by the Planck law and is, therfore, isotropic. Then the intensity of the radiation at a perpendicular distance x from the trailing end cap is given by

$$i_\nu(x) = \left(\frac{\nu}{\nu_e}\right)^3 i_{\nu_e} \tag{3.1}$$

where ν_e indicates the frequency of the photon emitted from the trailing end cap and ν the frequency of that same photon at the point x. It is straightforward to show that in the instantaneously co-moving frame these two frequencies are related by

$$\nu \approx \nu_e \left(1 - \frac{ax}{c^2}\right) \tag{3.2}$$

where a is the acceleration of the cavity. The relation is valid regardless of where on the end cap the photon originated. (This is the equivalent of gravitation redshift to which we will return in Section 3.2.) Thus

$$i_\nu(x) = \left(1 - \frac{ax}{c^2}\right)^3 i_{\nu_e}. \tag{3.3}$$

Because i_{ν_e} is given by the Planck function and therefore independent of direction, the implication is that i_ν is also isotropic. Of course, one must consider the Doppler shifted photons emitted from the leading encap and these photons are blue shifted. It turns out that in order to be in thermal equilibrium, the leading end cap must be at a lower temperature than the trailing end cap with the result that the intensity of photons emitted from the leading end cap is precisely the same as that of the photons emitted from the trailing end cap. (This argument will be elaborated on in Section 3.2.) The result is that, at least to first order in ax/c^2, the radiation in the cavity is istropic.

In this case, we can again use the perfect-fluid form of the stress-energy tensor Eq. (2.23) to describe the radiation. From conservation of energy/momentum we know that in any inertial frame $T^{\mu\nu}{}_{,\nu} = 0$ within the cavity. The spatial part ($\mu = i$) of this relation can be expressed in terms of the pressure, energy density and ordinary vector velocity $\mathbf{v}$ as [Weinberg 1972]

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{c^2(1-\beta^2)}{(\rho + p)} \left[\nabla p + \frac{\mathbf{v}}{c^2}\frac{\partial p}{\partial t}\right] \tag{3.4}$$

where $\mathbf{v}$ is the velocity of the cavity in the lab frame. (In all that follows, we refrain from distinguishing primed and unprimed frames since all calculations will be carried out in the inertial laboratory frame.) Because the cavity is assumed to be in equilibrium, the co-moving inertial frame pressue p and density ρ are independent of time. In addition, for small velocities we can discard terms that are second order in β^2 and the x component of this relation becomes

$$\frac{\partial p}{\partial x} = -\frac{(\rho + p)}{c^2} = -\frac{4pa}{c^2}. \tag{3.5}$$

To first order in ax/c^2, the solution to Eq. (3.5) is

$$p = p_0 \left(1 - \frac{4xa}{c^2}\right) = p_0 - \frac{4}{3c^2}\rho_0 ax \tag{3.6}$$

where p_0 and ρ_0 are the radiation pressure and energy density at the trailing end of the cavity. Finally, the forces that must be applied to the trailing and leading end caps of the cavity in order to maintain the acceleration must be $F_+ = p_0 A$ and $F_- = -p_0 A + (4AD\rho_0/3c^2)a$ where A is the area of each end cap and D is the length of the cavity. Therefore, the total force on the cavity must be

$$F = F_+ + F_- = \frac{4AD\rho_0}{3c^2}a = \frac{4}{3}\frac{E}{c^2}a \tag{3.7}$$

where E is, to lowest order, the radiation energy in the cavity co-moving frame.

We have made several assumptions in this derivation that need justification. First, we assumed that $Da/c^2 \ll 1$. This assumption is the requirement that the change in velocity of the cavity in one light crossing time is much less than the speed of light, the small acceleration condition. We have neglected any change in p and ρ in the transverse directions. The mirrored sidewall of the cylindrical cavity has the same effect on the radiation in the cavity as encaps of infinite transverse extent, in which case pressure and density only depend on x. The approximation that $E = \rho_0 AD$ neglects terms of order Da/c^2 but these only change the total force by terms that are second order in this quantity. In addition, we did not address what constitutes a constant acceleration of the cavity. Hasenöhrl's pre-relativistic scenario assumed a *rigid* cavity. In our calculation, we interpret the constant acceleration to be such that the cavity is *Born rigid*, that is, the cavity remains the same length in all instaneous co-moving intertial frames as would be expected for a cavity in an equilibrium state. Born rigid acceleration requires that the acceleration a_l of the leading end cap is related to the acceleration a_t of the trailing end cap by [Newman & Janis 1959]

$$a_l = \frac{a_t}{1 + a_t D/c^2}. \tag{3.8}$$

Therefore, the approximation that $a_l \approx a_t \approx a$ again neglects terms of order Da/c^2 and only changes the net force by terms second order in this quantity. Finally, we show in Section 3.2 that the fractional changes in temperature and pressure of blackbody radiation in an accelerated frame are of order aL/c^2 where L is the relevant dimension of the system in the direction of the acceleration. Therefore, one might expect that the deviation of a blackbody radiator (or an electromagnetic clock for that matter) in an accelerated frame would be of order aL/c^2 where L is some characteristic length of the process. For a blackbody radiator (or atomic clock) L might be the size of an atom, or the mean free length of a photon within the blackbody absorber, or perhaps the wavelength of the radiation. In any case, because L is much, much smaller than the size of the cavity, such effects are negligible. This provides justification for assuming that the blackbody end caps do, indeed, radiate according to Planck's Law.

If we identify the effective mass of the radiation in terms of $F = m_{\mathit{eff}}\, a$, then Eq. (3.7) implies that

$$m_{\mathit{eff}} = \frac{4}{3} E/c^2 \tag{3.9}$$

in agreement with our momentum analysis of Hasenöhrl's first gedanken experiment in Section 2.1 and with what Hasenöhrl found in his second gedanken experiment albeit using a conservation of energy argument. We can reproduce his energy argument result simply by integrating the net force in Eq. (3.7)

$$W = \int F v dt = \frac{4}{3}\frac{E}{c^2}\int a v dt = \frac{2}{3}E\beta^2, \tag{3.10}$$

which is precisely what Hasenöhrl found. Upon equating this work with kinetic energy of the radiation expressed as $m_{\mathit{eff}} = 1/2mv^2$, he found that the effective mass was that given by Eq. (3.9). On the other hand, our work/energy analysis of H1 found that $m_{\mathit{eff}} = \frac{5}{3}E/c^2$. Where have we (and Hasenöhrl) gone wrong?

Hasenöhrl was certainly familiar with Lorentz-Fitzgerald contraction and, in fact, invoked it in H2 and H3, although, not in his calculation of the work performed by the external forces. Because a Born rigid object has constant dimensions in instanteously co-moving frames, its length in the lab frame is Lorentz contracted. This is only approximately so. Because of their different accelerations, the velocities of the two ends of the cavity are not the same in the lab frame. Never the less, the usual expression for Lorentz contraction is valid to second order in β. Therefore, the distance moved by the leading end cap is less than that moved by the trailing end cap by an amount, $D - D/\gamma \approx \frac{1}{2}D\beta^2$. The work performed by the external forces in accelerating the cavity from rest to a velocity v is then

$$W = \int F_+ v_+ dt + \int F_- v_- dt = \int F_+ \Delta x_+ + \int F_- \Delta x_- \tag{3.11}$$

where $\Delta x_+ - \Delta x_- \approx \frac{1}{2}D\beta^2$. Substituting the expression for F_+ and F_- from above we find,

$$W = \frac{4AD\rho_0}{3c^2} a\Delta x_- + Ap_0(\Delta x_+ - \Delta x_-) = \frac{4AD\rho_0}{3c^2} a\Delta x_- + Ap_0 \frac{1}{2} D\beta^2. \tag{3.12}$$

Now $\Delta x_+ = \Delta x_- + O(\beta^2)$ and the displacement Δx is related to the velocity v and acceleration a by $v^2 \approx 2a\Delta x$. Also $Ap_0 D \approx E/3$ and $AD\rho_0 \approx E$. Thus the net work performed by the forces in the lab frame is

$$W = \frac{5}{6} E\beta^2. \tag{3.13}$$

Setting this equal to $\frac{1}{2}m_{\mathit{eff}} v^2$ gives $m_{\mathit{eff}} = \frac{5}{3}E/c^2$, precisely the same result as we got from our conservation of energy analysis of Hasenöhrl's first gedanken experiment.

So it seems that a proper analysis of Hasenöhrl's two gedanken experiments give consistent results and are also consistent with the relatistic expressions for energy and momentum of blackbody radiation. The problem is that the results from energy conservation imply an effective mass that is different from that implied by conservation of momentum and both of these are different from the $m_{\mathit{eff}} = E/c^2$ that we are led to expect from special relativity. This dilema is closely associated with a similar situation for classical models of the electron and we return to these issues in Section 4. First, however, we consider an analogous situation of a blackbody cavity at rest in a uniform, static gravitational field.

3.2 Blackbody cavity in a static gravitational field

Suppose the cylindrical blackbody cavity is at rest in a static, uniform gravitational field with the axis of the cavity in the direction of the field. We again use Liouville's theorem, Eq. (3.1), this time in combination with the usual equation for the gravitational redshift of photons, i.e.,

$$\nu \approx \nu_e \left(1 - \frac{gx}{c^2}\right) \tag{3.14}$$

where ν_e is the frequency of a photon emitted from the bottom end cap, ν is the frequency of that same photon at a height x above the bottom end cap, and g is the local acceleration of gravity. Of course, by the equivalence principle, this expression is the same as Eq. (3.2) with $g = a$. Combining Eqs. (3.1) and (3.14) again yields Eq. (3.3) with $a = g$, i.e., the upward intensity of radiation at point x due to photons emitted from the bottom end cap. The upward directed intensity of the radiation incident on the top end cap $i_\nu(D)$ due to the intensity of radiation emitted by the bottom end cap $i_{\nu_e}(0)$ is given by

$$i_\nu(D) = \left(1 - \frac{gD}{c^2}\right)^3 i_{\nu_e}(0). \tag{3.15}$$

The total flux incident on the upper end cap is

$$f_u = \int\int i_\nu(D) d\nu cos(\theta) d\Omega = \left(1 - \frac{gD}{c^2}\right)^4 \int i_{\nu_e}(0) d\nu_e \int cos(\theta) d\Omega. \qquad (3.16)$$

The integrals on the right hand side of this equation are well known and their product is given by $\sigma T(0)^4$ where $T(0)$ is the temperature of the lower end cap and σ is the Stefan-Boltzmann constant. On the other hand, the flux emitted by the upper blackbody end cap is the usual $\sigma T(D)^4$. These two fluxes must be equal if the system is in equilibrium, thus $T(D) = (1 - gD/c^2)T(0)$. Using this relation, it is straightforward to show that the downward intensity at an interior point x due to photons emitted from the upper end cap is equal to the upward intensity of the photons emitted from the lower end cap at the same point, as was asumed in Section 3.1.

In fact, it is easily demonstrated that the radiation at any point x in the interior of the cavity has a blackbody spectrum characterized by a temperature $T(x) = (1 - gx/c^2)T(0)$. From the Planck formula we know that the phase space density of blackbody radiation is

$$\frac{i_\nu}{\nu^3} \propto \left(1 + e^{\frac{h\nu}{kT}}\right)^{-1}. \qquad (3.17)$$

We assume that the radiation emitted from the bottom end cap has a blackbody spectrum and, therefore, obeys this relation. By Liouville's theorem, the phase space density of the radiation at point x, is equal to that of the emitted radiation, i.e.,

$$\frac{i_\nu}{\nu^3} = \frac{i_{\nu_e}}{\nu_e^3} \propto \left(1 + e^{\frac{h\nu_e}{kT(0)}}\right)^{-1} = \left(1 + e^{\frac{h\nu}{kT(x)}}\right)^{-1} \qquad (3.18)$$

where from Eq. (3.14) $T(x) \equiv (1 - gx/c^2)T(0)$. Therefore, the radiation at x also has a blackbody spectrum with a characteristic temperature $T(x)$. One can easily demonstrate that the same result is obtained by considering the gravitational blueshifted photons emitted from the top end cap.

Now imagine that the top and bottom end caps of the cavity are held in place not by internal cavity stresses but rather by external forces, i.e., the end caps are otherwise free to slide up and down inside the cavity. The radiation pressure pushing down on the lower end cap is $p(0) = \rho(0)/3 \propto T(0)^4$ while the pressue pushing up on the upper end cap is $p(D) = \rho(D)/3 \propto T(D)^4$. Therefore, $p(D) = (1 - aD/c^2)^4 p(0)$. The force required to support the bottom end cap is its weight, $M_{ec}g$, plus the force required to balance the pressure, $p(0)A$ and the force required to support the top end cap is clearly $M_{ec}g - p(D)A$. In addition, of course, a force $M_{sw}g$ is needed to support the side wall of the cavity. Finally the total force required to support the entire cavity, including radiation, is

$$F \approx Mg + \frac{4ADp(0)}{c^2} g \approx \left(M + \frac{4}{3}\frac{E}{c^2}\right) g \qquad (3.19)$$

where $M = 2M_{ec} + M_{sw}$ is the total mass of the cavity. It is clear from this expression that the weight of the radiation, $m_{eff}\, g$, implies an effective mass of $\frac{4}{3}E/c^2$, the same as Hasenöhrl deduced and consistent with the results of our momentum analyses for both of Hasenöhrl's gedanken experiments. In the gravitational case there is no work performed by the external forces and, hence, no analog of our work/energy analyses. Eq. (3.19) also justifies neglecting the mass of the cavity in Section 3.1. Again, we find a result that seems to contradict Einstein's $E = mc^2$.

This seeming contradiction and the connection with similar results for the classical electron brings us to a more general discussion of the energy and momentum of extended objects.

4 Hasenöhrl, Fermi, and the classical model of the electron

In Sections 2 and 3 we found momentum convservation and energy conservation in Hasenöhrl's two gedanken experiments led to two different effective masses associated with blackbody radiation, $m_{eff} = \frac{4}{3}E/c^2$ and $m_{eff} = \frac{5}{3}E/c^2$. Futhermore, these two masses were found to agree with the standard expressions for energy and momentum, i.e., $E = \int T^{00} dV$ and $P^i = \frac{1}{c}\int T^{0i} dV$. That these two expressions lead to different effective masses is a direct consequence of the integrals not being Lorentz covariant, i.e., E and $\mathbf{P}$ do not constitute a covariant 4-vector. If they did, it is straightforward to show that the expressions for both energy and momentum would imply an effective mass of E_0/c^2, the Einstein relation. Suppose $(E, \mathbf{P})$ is an energy/momentum 4-vector. In the zero momentum frame this is $(E_0, 0)$. A Lorentz boost to a frame with velocity $-\mathbf{v}$ immediately gives

$$E = \gamma E_0 \approx E_0 + \frac{1}{2}E_0\beta^2 = E_0 + \frac{1}{2}\frac{E_0}{c^2}v^2 \tag{4.1}$$

and

$$\mathbf{P} = \gamma E_0 \frac{\mathbf{v}}{c^2} \approx \frac{E_0}{c^2}\mathbf{v}. \tag{4.2}$$

If one identifies the kinetic energy, $E - E_0$, with $\frac{1}{2}m_{eff}v^2$ and the momentum with $m_{eff}\mathbf{v}$, both of these relations imply $m_{eff} = E_0/c^2$. A solution to the delimma might be simply to redefine the total energy and momentum of an extended body, in this case blackbody radiation, so that they are the components of a 4-vector. On the otherhand, the two Hasenöhrl gedanken experiments present the same dilemma and these results are derived from the work/energy theorem and conservation of momentum, neither of which seems amenable to redefinition.

A similar situation occurs in the case for the energy and momentum of the electromagnetic field surrounding a charged spherical shell (the classical electron). It is straightforward to show that the integral expressions for energy and momentum give precisely the same two results as those for blackbody radiation, i.e., Eqs. (2.28) and (2.30). Perhaps because most analyses make use of Newton's 2nd law/momentum conservation, historically such analyses deduced that $m_{eff} = \frac{4}{3}E/c^2$, hence, the "$\frac{4}{3}$ problem". One of the controversial issues is whether or not one must take into account the forces needed to make stable the repulsive charge of the electron. Poincaré (1906) was the first to consider the stability of the electron and introduced "Poincaré stresses," which were unidentified nonelectromagnetic stresses meant to bind the electron together. With the inclusion of these stresses, one finds that the effective mass of the electron is, indeed, $m_{eff} = E/c^2$ if one includes in E the contribution of Poincaré stresses. (Poincaré suggested more than one model for stabilizing stress [Cuvaj 1968].)

Max von Laue (1911) was the first to generalize this conclusion. He demonstrated that for any closed, static (extended) system for which energy and momentum are conserved, i.e., $T^{\mu\nu}{}_{,\nu} = 0$, the energy and momentum computed according to

$$P^{\mu} = \int T^{0\mu} dV \tag{4.3}$$

do indeed comprise a 4-vector. Felix Klein (1918) extended Laue's proof to time-dependent, closed systems. The conclusion is that for any closed, conservative system the *total* energy/momentum, defined by Eq. (4.3), is a 4-vector and, as a consequence, $m_{eff} = E_0/c^2$. (For a simple version of Klein's proof, see [Ohanian 2012].) As a consequence of Klein's theorem, it follows that the 4-momentum P^{μ} is related to the 4-velocity u^{μ} of the zero momentum frame center of mass (center of energy) by

$P^\mu = (E_0/c^2)u^\mu$ (e.g., [Møller 1972]). It is then straightfoward to show that, for any time-dependent, closed system, $\mathbf{F} = \gamma(E_0/c^2)\mathbf{a}$ where E_0 is the total energy in the zero momentum frame, $\mathbf{a}$ is the acceleration of the zero momentum frame center of mass, and $\mathbf{F}$ is the external force on the otherwise conservative system.

At first blush, the theorems of Laue and Klein might seem to contradict our results for Hasenöhrl's two gedanken experiments; however, neither of these satisfy the Laue/Klein assumption that the system is closed. For the Hasenöhrl scenarios, external forces (not included in $T^{\mu\nu}$) are necessary to contain the radiation. If instead, the radiation is contained by stresses in the cavity walls *and* these stresses are included in $T^{\mu\nu}$, then it is straightforward to show that the total energy and momentum from Eq. (4.3) are consistent with $m_{\mathit{eff}} = E/c^2$ where E is the total energy of the radiation plus cavity. Hasenöhrl certainly supposed that the radiation was contained by the cavity; however, he chose to consider the forces due to cavity stresses as *external*. This is a legitimate and understandable point of view. After all, Hasenöhrl was interested in the inertial mass of blackbody radiation, not the combined inertial masses of the radiation plus cavity.

In two of his earliest papers, Fermi (1922 & 1923a) took another approach to solving the $\frac{4}{3}$ problem, one that made no mention of the Poincaré stresses necessary to stabilize the electron. Fermi maintained that the $\frac{4}{3}$ problem for the classical electron arises because the electron is assumed to be a rigid body, in contradiction to the principles of special relativity. He applied the concept of "Born rigidity" to the electron, which requires that given points in an object always maintain the same separation in a sequence of inertial frames co-moving with the electron. Equivalently, Born rigidity demands that the worldline of each point in the electron should be orthogonal (in the Lorentzian sense) to constant-time hypersurfaces in the co-moving frames (see, eg., Pauli 1921). However, such constant-time hypersurfaces are of course not parallel to those in the lab. A constant-time integration over the electron's volume in its rest frame assumes that two points on the electron's diameter cross the $t = 0$ spatial hypersurface simultaneously, but this will *not* be the case in a Lorentz-boosted frame [Boughn & Rothman 2011]. Fermi chose to evaluate the action by integrating over the volume contained within the constant-time hypersurfaces in the co-moving frame (equivalent to using Fermi normal coordinates, which he developed in an earlier paper [Fermi 1923a]). In a sense, this choice renders the analysis covariant, i.e., independent of the lab frame, and it is, perhaps, not surprising that the result of his analysis is that $\mathbf{F} = (E/c^2)\mathbf{a}$. The details of Fermi's approach can be found in Jackson [1975] and Bini [2011]. It should be emphasized that Fermi's solution to the $\frac{4}{3}$ problem, unlike the Poincaré/Laue/Klein approach, is silent on any non-electromagnetic forces that hold the electron together. Like Fermi, Fritz Rohrlich also sought to solve the $\frac{4}{3}$ problem without addressing the stability of the electron. Rohrlich [1960] simply redefined the expression for total energy/momentum in Eq. (4.3) so that it is covariant and, thus, constitutes a proper 4-vector.

In a second 1923 paper [Fermi & Pontremoli 1923b], Fermi and Pontremoli applied the above prescription to solve Hasenöhrl's cavity-radiation problem. They considered the forces applied to a volume of radiation and restricted their attention to the slowly accelerated case. Therefore, their results apply to Hasenöhrl's second gedanken experiment. They concluded that the acceleration of the radiation in the cavity requires a force $\mathbf{F} = (E/c^2)\mathbf{a}$ independent of any forces (e.g., cavity stresses) that contain the radiation. While it might seem that Fermi's and Rohlich's insistance on a covariant approach is a reasonable demand, the resulting analyses do not seem to be capable of capturing the physics of a Hasenöhrl-type problem. This should give one pause.

Whether the Fermi/Rohrlich approach or that of Poincaré, von Laue, and Klein is the appropriate description of the classical electron remains a controversial subject and continues to foster arguments on both sides of the issues. A sample over the last 50 years includes papers by: Rohrlich [Rohrlich 1960; Rohrlich 1982]; Gamba [Gamba 1967]; Boyer [Boyer 1982]; Campos and Jiménez [Campos & Jiménez 1986]; Campos [Campos et al. 2008]; and Bini et al. [Bini 2011]. The second edition of Jackson's *Classical Electrodynamics* [Jackson 1975] discusses both approaches. The interested reader is referred to these works. With regard to classical models of the electron, both methods give the same result and the electron is, in any case, fundamentally a quantum phenomenon.

On the other hand, these issues are not ambiguous in the case of Hasenöhrl's blackbody cavity. In this case, neither of the approaches of the two schools is particularly helpful. The Laue/Klein theorem cannot be invoked because the system is not closed; the forces that contain the radiation are external to the system. We suspect that that members of the Laue school would agree with this point of view. (Of course, if a blackbody cavity is stabilized by stresses within the cavity walls, then the Laue/Klein theorem would indeed apply with the result that $\mathbf{F} = (E/c^2)\mathbf{a}$ where E is the total rest frame energy of the radiation *and* cavity.) On the other hand, Fermi's own analysis of Hasenöhrl's slowly accelerating blackbody cavity yields a result in conflict with our relativistic analysis. One suspects that precisely the same would be true for a macroscopic charged spherical shell with the charge held in place by external forces. (We plan to analyze this system elsewhere.)

We refrain from taking a point of view on the controversy regarding the structure of the fundamentally quantum mechanical electron nor even will we argue that the Fermi/Rohrlich definition of relativistic energy/momentum is invalid. On the contrary, its covariant nature has a certain appeal. However, it is clear that the ideological application of this notion without regard to the details of a system can lead one astray. In particular, identifying E/c^2 with the effective mass of blackbody radiation leads immediately to $\mathbf{F} = (E/c^2)\mathbf{a}$ for Hasenöhrl's slowly accelerating cavity, which is in conflict with a proper relativistic analysis. One might argue that systems bound by external forces rarely occur in problems dealing with relativistic mechanics. This may be true; however, the purpose of Hasenöhrl's gedanken experiment, and Fermi's response for that matter, was to answer foundational problems in physics. In this sense Hasenöhrl's $m_{\mathit{eff}} = \frac{4}{3}E/c^2$ is correct and Fermi's $m_{\mathit{eff}} = E/c^2$ appears to be wrong.

One might argue that Fermi's analysis, while not explicitly including the forces necessary to contain the radiation, might finesse the problem by assuming Born rigidity. On the other hand, our relativistic analysis also assumes Born rigidity and yet arrives at a different result. Another possibility is that Fermi's analysis somehow only includes that part of the external force necessary to accelerate the radiation and ignores that part of the force that stabilizes the cavity; however, how one might effect such a separation of forces is not immediately obvious. Of course, it is possible that Fermi simply misunderstood what Hasenöhrl meant by "external forces". Perhaps the important lesson of this exercise is that while $E = mc^2$ is a ubiquitous and very valuable relation, it is not a "law of physics" that can be used indiscriminately without regard to the details of the system to which it is applied.

It is often claimed that Einstein's derivation of $E = mc^2$ was the first generic proof of the equivalence of mass and energy (see Ohanian [2009] for arguments to the contrary). It is true that Hasenöhrl's analysis was restricted to the inertial mass of blackbody radiation; however, Einstein's gedanken experiment involves radiation emitted from a point mass and, futhermore, gives no indication how this occurs. If it is radiation due to radioactive decay, as Einstein implies at the end of his paper, then perhaps it is necessary to take into account the details of this process. In any case, Einstein is clearly speaking about electromagnetic radiation, and so it is difficult to conclude that his thought experiment should be taken as a general theorem about mass and energy. Einstein's great contribution was, perhaps, that based on his simple gedanken experiment, he *conjectured* that $E = mc^2$ was broadly true for all interactions. Over time, his conjecture was justified theoretically and verified experimentally, but this was through the efforts of many scientists and engineers.

Fritz Hasenöhrl attempted a legitimate thought experiment and his analysis, though hampered by a pre-relativistic world view, was certainly recognized as significant at the time. Whether or not his analysis was completely consistent, one of his conclusions, that the acceleration of blackbody radiation by external forces satisfies $\mathbf{F} = \frac{4}{3}(E/c^2)\mathbf{a}$, was correct, even if of limited applicability, and for this he should be given credit. In addition, his gedanken experiment raises similar questions for the classical electron, issues that remain of interest today. Hasenöhrl's gedanken experiments are worthy of study and are capable of revealing yet another of the seemingly endless reservoir of the fascinating consequences of special relativity.

Acknowledgements. I thank Tony Rothman for introducing me to this fascinating problem and for contributing much to this paper. Our previous joint work [Boughn & Rothman 2011] includes some of the content of the present paper and, in addition, discusses much more of the history of $E = mc^2$. I am grateful to Bob Jantzen for sharing [Bini 2011] before publication, and I thank him as well for a helpful conversation. Thanks are due Jim Peebles for helpful conversations and expecially for pointing out the relevance of Liouville's theorem, and to Hans Ohanian for criticisms that prompted a substantial change in the analysis of Hasenöhrl's second gedanken experiment.

References

Bini, D., A. Geralico, R. Jantzen and R. Ruffini. 2011. On Fermi's resolution of the '$\frac{4}{3}$ problem' in the classical theory of the electron: hidden in plain sight. To appear in *Fermi and Astrophysics*, edited by R. Ruffini and D. Boccaletti. World Scientific, Singapore, 2011

Boughn, S. and T. Rothman. 2011. Hasenöhrl and the Equivalence of Mass and Energy. `arXiv:1108.2250`

Boyer, T. 1982. Classical model of the electron and the definition of electromagnetic field momentum. *Phys. Rev.* D **25**: 3246-3250

Campos, I. and J. Jiménez. 1986. Comment on the $\frac{4}{3}$ problem in the electromagnetic mass and the Boyer-Rohrlich controversy. *Phys. Rev.* D **33**: 607-610

Campos, I., Jiménez, J. and Roa-Neri, J. 2008. Comment on "The electromagnetic mass in the Born-Infeld theory". *Eur. J. Phys.* **29**: L7-L11

Cuvaj, C. 1968. Henri Poincaré's Mathematical Contributions to Relativity and the Poincaré Stresses. *Am. J. Phys.* **36**: 1102-1113

Fermi, E. 1922. Correzione di una contraddizione tra la teoria elettrodinamica e quella relativistica delle masse elettromenetiche. *Nuovo Cimento* **25**: 159-170. English translation to appear as "Correction of a contradiction between the electrodynamic theory and relativistic theory of electromagnetic masses," in *Fermi and Astrophysics*, edited by R. Ruffini and D. Boccaletti. World Scientific, Singapore, 2012

Fermi, E. 1923a. Sopra i fenomena che avvengono in vicinanza di una linea oraria. *Rend. Lincei* **31**: 21-23. English translation to appear as "On phenomena occuring close to a world line," in *Fermi and Astrophysics*, edited by R. Ruffini and D. Boccaletti. World Scientific, Singapore, 2012

Fermi, E. and A. Pontremoli. 1923b. Sulla mass della radiazione in uno spazio vuoto. *Rend. Lincei* **32**: 162-164. English translation to appear as "On the mass of radiation in an empty space," in *Fermi and Astrophysics*, edited by R. Ruffini and D. Boccaletti. World Scientific, Singapore, 2012

Gamba, A. 1967. Physical quantities in different reference systems according to relativity. *Am. J. Phys.* **35**: 83-89

Hasenöhrl, F. 1904a. Zur Theorie der Strahlung in bewegten Körpern. *Wiener Sitzungsberichte* **113**: 1039-1055

Hasenöhrl, F. 1904b. Zur Theorie der Strahlung in bewegten Körpern. *Annalen der Physik* **320**: 344-370

Hasenöhrl, F. 1905. Zur Theorie der Strahlung in bewegten Körpern, Berichtigung. *Annalen der Physik* **321**: 589-592

Hasenöhrl, F. 1907, 1908. Zur Thermodynamik bewegter Systeme. *Wiener Sitzungsberichte* **116**, IIa (9): 1391-1405 and **117**, IIa (2): 207-215

Jackson, J. 1975. *Classical Electrodynamics*, 2nd edn. John Wiley and Sons, New York

Jammer, M. 1951. *Concepts of Mass*. Harvard University Press, Cambridge

Jammer, M. 2000. *Concepts of Mass in Contemporary Physics and Philosophy*, pp. 72–73. Princeton University Press, Princeton

Klein, F. 1918. Über die Integralform der Erhaltungssätze und der Theorie die räumlich-geschlossenen Welt. *Nach. Gesell. Wissensch. Göttingen, Math.-Physik, Klasse*, 394-423

Laue, M. 1911. *Das Relativitätsprinzip*. Vieweg, Braunschweig

Misner, C., K. Thorne and J. Wheeler. 1973. *Gravitation*. W.H. Freeman, New York

Møller, C. 1972. *The Theory of Relativity*. Oxford University Press, Oxford

Newman, E., and A. Janis. 1959. Ericksen, E. et al. 1982. Rigid Frames in Relativity. Relativistic rigid motion in one dimension. *Phys. Rev.* **116**: 1610-1614

Ohanian, H. 2009. Did Einstein Prove $E = mc^2$? *Studies in History and Philosophy of Modern Physics* **40**: 167-173

Ohanian, H. 2012. Klein's Theorem and the Proof of $E = mc^2$. *Am. J. Phys.*, in press

Pauli, W. 1921. *Theory of Relativity*. Pergamon Press, London, 1958

Peebles, J. and D. Wilkinson. 1968. Comment on the anisotropy of the primeval fireball. *Physical Review* **174**: 2168

Poincaré, H. 1906. Sur la dynamic de l'electron. *Rendiconti del Circolo matematico di Palermo* **21**: 129176

Rohrlich, F. 1960. Self-energy and stability of the classical electron. *Am. J. Phys.* **28**: 639-643

Rohrlich, F. 1982. Comment on the preceeding paper by T.H. Boyer. *Phys. Rev.* D **25**: 3251-3255

Thomson, J.J. 1881. On the electric and magnetic effects produced by the motion of electrified bodies. *Philosophical Magazine* **11**: 229-249

Weinberg, S. 1972. *Gravitation and Cosmology*. John Wiley & Sons, New York